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THE COMPLETENESS OF FORMAL SYSTEMS

by

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ABSTRACT

The problem of the completeness of formal systems is re-examined and a new definition suggested which agrees with older concepts for such simple systems as had been provided with completeness proofs, but differs for more complex logics where only negative results had been encountered. In terms of the new notion completeness proofs are developed for a wide class of formal systems.

Several applications of the methods used in these proofs are made to problems in algebra, including a very short demonstration of the representation theorem for Boolean rings. A discussion of some implications of the principal results for set- and number- theory makes up the last section.

AN EXPRESSION OF APPRECIATION

It is a pleasure to record here my thanks to Professor Alonzo Church, not only for encouragement, suggestion, and criticism in connection with the formation of this dissertation but equally for many insights into the nature of Logic and the development of a point of view toward this subject matter without which most research work must founder.

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INTRODUCTION

From one point of view formal logic is the application of mathematics to the analysis of the way in which symbols are used in scientific discourse. When the language of mathematics itself becomes the object of mathematical investigation, we can expect three general types of result:

1. Insight into the foundations and philosophy of mathematics;

2. Discovery of limitations imposed by the nature of mathematical tools on the kinds of problem which can be stated or proved; and

3. The development of new and improved methods to strengthen the attack on problems in the various branches of mathematics.

Historically, problems of the first category were principally responsible for the development of modern logic, and almost all work in the field has some aspect bearing on the logical basis of science. Important contributions to this domain are associated with the names of Boole, Frege, Russell, Brouwer, Hilbert, Carnap, and Tarski. Although there are still important problems in foundations which remain open for investigation and debate, this field is the locus of greatest success for formal methods.

Results of the second type were not foreseen during the early growth of the subject. They came, somewhat unexpectedly, as an outgrowth of and commentary upon formal techniques developed originally for much more innocent purposes. The important results of this category are those of Skolem, Godel, and Church.

As for the third class of results, it must be admitted that these

are most meager. Whereas the methods and results of algebra, geometry and analysis are richly intertwined, mathematical logic has held a rather isolated position within the family of mathematical disciplines. The dream of people like Leibniz, Mill, and Hilbert of a universal calculus of reasoning which would provide a mechanical solution to any mathematical problem appears definitely shattered by the results of Church and Gödel. As for more modest contributions to the various branches of mathematics, Boole's original development of the algebraic structures now named after him is perhaps as prominent an example as any.

In this paper we present results intersecting each of the three categories mentioned above. As central problem we consider the completeness of formal systems, which is principally a foundational question. Roughly, we seek to show that the formal axioms and recursive rules of inference of each of various logics are adequate to prove all those formulas which express true propositions when the system is given its intended interpretation.

In Part I we develop a method which yields new proofs of the completeness of the propositional and first order functional calculus, results originally obtained by Post and Gödel, respectively. A new set of axioms is given for the former. These results are combined and cast in a strong form from which, as a corollary, we derive the Skolem-Löwenheim result that if a set of formulas of the functional calculus of first order is satisfiable by some model then it can be satisfied by a denumerable model.

In Part III we consider a system for the functional calculus of

order ω based upon Church's formulation of the simple theory of types. This is a system sufficiently rich in vocabulary to come under the scope of Gödel's results on undecidable statements, results which have often been interpreted as constituting a proof of the incompleteness of those systems to which they apply. However, we point out that for these systems there is a wide class of models which provide an interpretation under which provable formulas are true; we recast the definition of completeness to take account of these models, and extend the method of Part I to obtain a completeness proof for the general functional calculus.

Our remarks on the limitations of the mathematical method are concentrated in Part IV. We show that the existence of models for mathematical systems, other than those intended in setting up these systems, is characteristic not only of the pure functional calculi but of applied systems such as may be obtained, for example, by adding Peano's axioms for the natural numbers. By this means we obtain stronger forms of Skolem's results on the impossibility of characterising the natural numbers, and at the same time throw light on the matter of ω-consistency first investigated by Tarski and Gödel.

The possibility of employing the techniques of symbolic logic in other branches of mathematics are emphasized in Part II, where we give two applications of the ideas of Part I to algebra. In the first we obtain a very general imbedding theorem for algebraic structures which yields, as special cases, the representation theorems for Boolean algebras and distributive lattices originally obtained by Marshall Stone and Garrett Birkhoff through the use of ideal theory.

As a second application we describe a class of results which may be carried over directly from modular fields to fields of characteristic zero.

Our conjecture is that the potential aid (of the type indicated in Part II) which mathematical logic can render to older branches of mathematics is abundant and accessible, and that a rich harvest lies just below the surface.

PART I

THE FIRST ORDER FUNCTIONAL CALCULI

The functional calculi of first order are designed to discuss, to a certain limited extent, three types of domain: a domain, O consisting of the two truth values T and F; an arbitrary domain of individuals, $\mathcal{I}$; and for each n = 1, 2, 3, ... a domain, C_n consisting of functions of n variables ranging over $\mathcal{I}$, with values in O. The elements of C_n may be identified in an obvious way with classes of ordered n-tuples of individuals, or with n-ary relations holding among individuals.

The pure functional calculus of first order, which we shall now set up for investigation, will contain the following primitive symbols as the basis of its vocabulary.

An infinite set of propositional variables:

$$p\ q\ r\ s\ p'\ q'\ r'\ s'\ p''\ \ldots$$

An infinite list of individual variables:

$$u\ v\ x\ y\ z\ u'\ v'\ x'\ y'\ z'\ u''\ \ldots$$

An infinite list of functional variables of degree n (n = 1, 2, 3, ...):

$$F_n\ G_n\ H_n\ F_n'\ G_n'\ H_n'\ F_n''\ \ldots$$

A single propositional constant:

$$f$$

Improper symbols:

$$(\ ),\ \supset$$

The primitive symbols of various applied functional calculi which we will have occasion to consider later will differ from the list given above by the possible addition of individual and functional constants and further improper symbols, and the possible omission of propositional and functional (but not individual) variables.

Certain finite sequences of primitive symbols are used as sentences (well-formed formulas; w.f.f.) to describe situations

occurring in the domains 0, $\mathfrak{I}$, and C_n. These formulas are defined by the following recursive rules.

A propositional variable or constant alone is a well-formed formula.

If C is a functional variable or constant of degree n and $a_1, a_2, \ldots, a_n$ are individual variables or constants then $C(a_1, a_2, \ldots, a_n)$ is a well-formed formula.

If A and B are w.f.f. so are $(A \supset B)$ and $(a)A$, where a is any individual variable.

No formula is w.f. unless the above rules compel it to be.

As illustrated above, we use letters 'A', 'B', 'C' as syntactical variables ranging over w.f.f. These letters are not part of the vocabulary of the functional calculi; but are used as an aid in talking about these formal systems. Similarly we use letters 'a', 'b', and 'c' with various subscripts as syntactical variables ranging over the several kinds of constants and variables of the calculi, while the improper symbols and juxtaposition are used to denote themselves in an obvious and contextually unambiguous manner.

Definition: An occurrence of an individual variable, a, in a w.f.f., A, is a <u>bound occurrence</u> if it is in a w.f. part of A of the form $(a)B$; otherwise it is a <u>free occurrence</u>.

We now explain how the w.f.f.'s are to be interpreted; that is, how they are to be used as sentences to express propositions about domains 0, $\mathfrak{I}$, and C_n.

A domain of individuals, $\mathfrak{D}$, may be referred to as a model.

A valuation of a w.f.f. A with respect to a given model consists in an assignment as follows: To each propositional variable or constant of A assign an element of $\mathfrak{D}$; i.e. a truth-value, T or F; to each individual constant or individual variable with free occurrences in A assign an element of $\mathfrak{J}$; to each functional variable or constant of degree n in A assign an element of $\mathfrak{C}_n$; i.e. a truth-valued function of n arguments each of whose range is $\mathfrak{J}$.

A valuation is said to satisfy A with respect to a given model if A has the value T according to the following meanings of its basic components:

i. "f" denotes the truth value F (falsehood).

ii. $\supset$ is the relation of material implication, translated by one sense of the English phrase "if...then..."; "$A \supset B$" holds (has the value T) except when A has the value T and B has the value F.

iii. (a) is the universal quantifier; "$(a)B$" means "B has the value T for every a".

iv. "$\mathfrak{C}(a_1, \ldots, a_n)$" means that the function $\mathfrak{C}$ has the value T for individuals $a_1, \ldots, a_n$ as arguments.

A w.f.f. A is satisfiable with respect to $\mathfrak{D}$ if some valuation with respect to $\mathfrak{D}$ satisfies A; is satisfiable if it is satisfiable with respect to some model.

A w.f.f. A is valid with respect to $\mathfrak{D}$ if every valuation with respect to $\mathfrak{D}$ satisfies A; A is valid if it is valid with respect to every model.

We may note that if a w.f.f. A is not valid then $A \supset f$ is satisfiable - i.e., to use an abbreviation introduced below, $\sim A$ is satisfiable.

Examples:

1. If φ is the element of C_2 assigned to "F_2", then this valuation satisfies

$$((x)(y)F_2^{11}(x,y) \supset f) \supset f)$$

with respect to some model, $\mathfrak{J}$, just in case there exists an x in $\mathfrak{J}$ such that for every y in $\mathfrak{J}$ $\varphi(x,y)$ is T.

2. The formula "$(x)F_1^1(x) \supset F_1^1(y)$" is valid.

The definition of the terms "valid" and "satisfiable", involving as it does the "meanings" of basic components of a formula, cannot be regarded as fully precise in the sense required by general mathematical practice. This precision can be attained by using the ideas of Tarski[1]; a short account of the details for the first order functional calculus can be found in Church[1], pp. 80-81. We shall not reproduce them here, since our imprecise description has sufficient intuitive clarity to provide a working basis for the results we shall seek. In Part III we will give precise definitions for the concepts of validity and satisfiability in a richer language than the present calculi.

In terms of the basic concepts expressed by the vocabulary of the functional calculi (as given on p. 3), we can define various other useful concepts. Formally, we embody these definitions in conventions which introduce new symbols for use in abbreviating well-formed formulas. These symbols are not a proper part of the formal calculus but merely dispensable aids for handling symbolic manipulations.

However, when we <u>interpret</u> formulas as expressing propositions about some model, or class of models, the symbols used in abbreviating take on derivative meaning related to the meaning of the primitive symbols in terms of which they are defined.

The more useful formal definitions are listed below, where the arrow is used to mean "stands for", or "is short for".

$\sim A \longrightarrow (A \supset f)$ (negation)

$A \vee B \longrightarrow \sim A \supset B$ (disjunction)

$A \cdot B \longrightarrow \sim(A \supset \sim B)$ (conjunction)

$(\exists a)B \longrightarrow \sim(a)\sim B$ (existential quantification)

$A \equiv B \longrightarrow (A \supset B) \cdot (B \supset A)$ (material equivalence)

In addition to these abbreviations we shall further facilitate formal manipulations by conventions as to the omission and restoration of parentheses, and the use of dots as an alternative method of bracketing. We take over in toto the rules concerning these conveniences which are given in Church[1], pp. 3, 4, 5, 39. The principal features of this system are the following rules for restoring parentheses to abbreviated formulas:

(a) A dot will replace an initial parenthesis, its mate being placed as far to the right as consistent with the formula's being well formed.

(b) Association to the left; e.g. "$p \supset q \supset r$" stands for "$((p \supset q) \supset r)$" rather than "$(p(q \supset r))$".

(c) Earlier symbols in the following list bind more closely than

later ones:

$(\exists a)\quad (a)\quad \sim\quad .\quad \vee\quad \supset$

Thus "$(a)F(x) \vee \sim P \cdot q$" stands for "$(((a)F(x)) \vee ((\sim P) \cdot q))$".

The rules for interpreting formulas enable us to obtain, in a step-by-step fashion, a necessary and sufficient condition for the validity of any given w.f.f. However, there is no effective method for deciding in each case whether these conditions hold. (It is clear that our definition of validity does not supply any such effective criterion; that there is in fact no way of characterizing the valid formulas which will provide a decision method was proved by Church[2].)

To enable us to recognize the validity of formulas in special cases we introduce the idea of a formal <u>proof</u>, starting with the following:

<u>Axiom schemata</u>. For any w.f.f.'s A, B, C:

1. $A \supset (B \supset A)$
2. $C \supset A \supset . C \supset (A \supset B) \supset . C \supset B$
3. $A \supset f \supset f \supset A$
4. $(a)A \supset B \supset . A \supset (a)B$, where a is any individual variable not free in A.
5. $(a)A \supset B$ where a is an individual variable, b an individual variable or constant, no free occurrence of a in A is in a w.f. part of A of the form $(\exists b)C$, and B is obtained by substituting b for all free occurrences of a in A.

<u>Rules of inference:</u>

I. From A and $A \supset B$ to infer B. (modus ponens).

II. From A to infer $(a)A$, where a is any individual variable. (generalization).

Definitions. A finite sequence of w.f.f.'s each of which is either an axiom or obtained from two or one earlier formulas of the sequence by an application of rule I or II is a (formal) proof. The last formula of a proof is a (formal) theorem. If A is a theorem we write "$\vdash A$".

If Λ is a set of w.f.f.'s, if A is a w.f.f. which is the last member of a proof in which the formulas of Λ as well as those of schemata 1-5 are counted as axioms, and if this proof involves no application of rule II upon a variable having free occurrences in some formula of Λ, we write "$\Lambda \vdash A$".

By checking the validity of formulas having each of the axiom-forms 1-5, and checking that applications of the rules of inference I and II preserve validity, we may easily verify that every formal theorem is valid. The problem of the completeness of the pure first order functional calculus is the question whether, conversely, every valid formula is formally provable from the given axioms by rules I and II.

The formal system for which a completeness proof was first obtained is the so-called propositional calculus, a fragment of the first-order functional calculus. One formulation of the propositional calculus may be obtained from the above account of the pure functional calculus by dropping all notation involving individual and functional variables and quantifiers, and retaining only axiom schemata 1-3 and Rule I. (The particular axiom system given here appears to be new,

although it is difficult to check this since most published axiom systems have appeared only in Polish papers.)

A valid formula of the propositional calculus is called a tautology. Unlike the situation prevailing in the case of the functional calculus there is an effective procedure to decide whether any given formula is a tautology.

The first proof that every tautology is a formal theorem of the propositional calculus was given by Post[1] in 1921. Simplified proofs were given by others at later dates, of which perhaps the shortest is Quine's[1]. A version of the latter appears in Church[1].

The first (and only) published proof of the completeness of the pure first order functional calculus was given by Gödel[1] in 1930. This proof does not furnish a new proof of the completeness of the propositional calculus, since the argument makes use of the latter fact.

In this section we shall give a new proof of the completeness of the pure functional calculus of first order. The proof is developed directly from the rules of the formal system, and hence the method yields (in a simplified version) a new proof of the completeness of the propositional calculus. Although the latter appears to be considerably shorter than previous proofs, these have the advantage of being constructive in a sense satisfactory to the Intuitionist school of mathematics while ours leans heavily on the principle of the excluded middle.

For the functional calculus of first order, Gödel's completeness proof and ours alike make use of the principle of excluded middle.

We begin by deriving some theorem schemata and a derived rule of inference.

6. $\vdash A \supset A$ (for each w.f.f. A).

Proof:

In (2) put $A \supset C$ for A, using the result together with an instance of 1. and Rule I to obtain $\vdash C \supset (A \supset C \supset B) \supset . C \supset B$.

In this put A for both C and B, using the result together with $\vdash A \supset . (A \supset A) \supset A$ (an instance of 1.) and rule I to obtain 6.

III (The "Deduction" Rule):

If $\Lambda, A \vdash B$, then $\Lambda \vdash A \supset B$ (for any w.f.f.'s A, B and any set, Λ, of w.f.f.'s)

Proof:

This theorem was first formulated by Herbrand in 1930. The proof, a simple induction using 1, 2, 4, 5, I and II, may be found in detail in Church[1], pp. 45-46, and we shall not reproduce it here. (The restriction in the published version to a finite set of "assumptions", Λ, is clearly unnecessary.)

7. $\vdash A \supset . B \supset f \supset . (A \supset B) \supset f$

Proof:

$A, A \supset B \vdash B$ by I.

$A, A \supset B, B \supset f \vdash f$ by I.

Hence 7. follows by three applications of III.

8. $\vdash f \supset A$

Proof:

$f \vdash (A \supset f) \supset f$ by 1, I

$f \vdash A$ by 3, I.

And 8. follows by applying III.

9. $\vdash A \supset f \supset . A \supset B$

Proof:

$A \supset f, A \vdash f$ by I

$A \supset f, A \vdash B$ by 8, I.

Applying III gives 9.

10. $\vdash (a)(A \supset f) \supset . (\exists a) A \supset f$

Proof:

$(a)(A \supset f), (a)(A \supset f) \supset f \vdash f$ by I

$\vdash (a)(A \supset f) \supset . \sim(a) \sim A \supset f$ by III (twice)

10. is an abbreviation of this.

11. $\vdash (a) A \supset f \supset . (\exists a)(A \supset f)$

Proof:

$(a)(A \supset f \supset f) \vdash A \supset f \supset f$ by 5, I.

$(a)(A \supset f \supset f) \vdash A$ by 3, I

$(a)(A \supset f \supset f) \vdash (a) A$ by II

$(a)(A \supset f \supset f), (a) A \supset f \vdash f.$ by I

$(a) A \supset f \supset . (a)(A \supset f \supset f) \supset f.$ by III (twice)

We now consider directly the problem of showing that all valid formulas are provable. For this purpose we consider a w.f.f. X which we suppose is not a theorem, and seek to determine a domain of individuals, $\mathfrak{J}$, and a valuation of X with respect to $\mathfrak{J}$ such that X has the value F for this valuation under the prescribed meanings for the basic components (see p.3).

As a simplification we may restrict our consideration to w.f.f.'s X such that X contains no free occurrence of any variable. For if X contains free variables we can consider X', the closure of X,

obtained from X by prefixing universal quantifiers so as to bind (in the order of their occurrence) each variable occurring free in X. From the definition of valid we see that X' is valid if and only if X is valid; while by repeated applications of 5 and I (II) we easily see that X is a theorem if (and only if) X' is a theorem.

Furthermore, to show that X is not valid it suffices to prove that $X \supset f$ is satisfiable.

Definition. A set, Λ, of (one or more) w.f.f.'s is called "inconsistent" if $\Lambda \vdash f$; otherwise Λ is consistent. From 8, I we see that any w.f.f. can be inferred from an inconsistent set.

Now if X is not a theorem then $\{X \supset f\}$ is a consistent set. For if $X \supset f \vdash f$ then by III $\vdash X \supset f \supset f$ and hence (3, I) $\vdash X$.

From the above remarks, therefore, we see that in order to demonstrate the completeness of our formulation of the functional calculus it suffices to prove that if K is a w.f.f. with no free variables such that $\{K\}$ is consistent, then K is satisfiable. In fact we shall prove a stronger theorem.

Definition. A set of formulas is simultaneously satisfiable if each formula is satisfiable in the same domain in such a way that the valuations satisfying the formulas of the set assign the same meaning (as an element of O, $\mathfrak{d}$, or C_h) to any given variable or constant in all of its occurrences in various formulas of the set.

Theorem I. If Λ is any consistent set of w.f.f.'s of the pure functional calculus (no member of which contains free variables), then the formulas of Λ are simultaneously satisfiable in a domain of

denumerably many individuals.

(This theorem was originally proved in Gödel[1], where the completeness is first proved, and it is then indicated how the method used there can be extended to obtain theorem I. We prove the theorem directly, and obtain completeness as a corollary.)

We start by fixing attention on a set of distinct symbols, u_{ij} $(i, j = 0, 1, 2\ldots)$, (different from any of the symbols appearing in the pure functional calculus of first order,) which we shall use as individual constants, as follows. Let us denote by "$\mathfrak{S}_0$" the pure functional calculus as formulated in the preceding pages. Let $\mathfrak{S}_1$ be the formal system obtained from $\mathfrak{S}_0$ by adjoining as new primitive symbols the individual constants u_{0j} $(j = 0, 1, 2, \ldots)$, the rules for forming w.f.f.'s, axioms and theorems being left unaltered. In the same way let $\mathfrak{S}_{i+1}$ be the system obtained from $\mathfrak{S}_i$ $(i = 1, 2, \ldots)$ by adjoining as new individual constants the symbols u_{ij} $(j = 0, 1, 2, \ldots)$. Finally, let $\mathfrak{S}_\omega$ be the system whose symbols are those of all the $\mathfrak{S}_i$ $(i = 0, 1, 2, \ldots)$ together.

For convenience in specifying a definite manner of making various choices in the following construction (thus avoiding use of the axiom of choice) we may suppose that distinct natural numbers have been assigned to the w.f.f.'s of $\mathfrak{S}_\omega$ (for example, by such a scheme as given in Gödel[2]) and refer to the first, second, ... formula of $\mathfrak{S}_\omega$ in the standard enumeration of all w.f.f.'s.

Now we start with $\mathfrak{S}_0$ and a given set, Λ, of w.f.f.'s of $\mathfrak{S}_0$ (containing no free variables) which is consistent. The first step of the construction involves obtaining a maximal consistent set Γ_0,

of w.f.f.'s of $\mathfrak{S}_0$ (containing no free variables) such that $\Lambda \subset \Gamma_0$ (i.e. all the formulas of Λ are also in Γ_0). This may be accomplished by considering one after another the formulas of $\mathfrak{S}_0$ (in their standard enumeration) and selecting or rejecting them according as they are or are not consistent with the formulas of Λ together with those previously selected and added to Λ. The set of formulas so selected may be taken as Γ_0; it will be consistent, since any proof of f would use only a finite number of its formulas; but such a proof would mean that at some stage of the selecting process directions were not followed. Γ_0 may be Λ itself; at any rate the addition to Γ_0 of any w.f.f. of $\mathfrak{S}_0$ (without free individual variables) produces an inconsistent set.

Concerning a set of formulas with the properties of Γ_0 we may make several remarks. In the first place any theorem (with no free individual variables) is clearly in Γ_0, and Γ_0 is closed under applications of the rules of inference (I and II). Secondly, if $\Gamma_0 \vdash A$ then $A \in \Gamma_0$. For since Γ_0 is <u>maximal</u> consistent, $A \notin \Gamma_0$ implies $\Gamma_0, A \vdash f$. Whence $\Gamma_0 \vdash A \supset f$ by III, which together with $\Gamma_0 \vdash A$ and I implies $\Gamma_0 \vdash f$, contrary to the consistency of Γ_0. Thirdly, for any w.f.f. A (in which all occurrences of individual variables are bound) either $\Gamma_0 \vdash A$ or $\Gamma_0 \vdash A \supset f$. For otherwise $\Gamma_0, A \vdash f$ and $\Gamma_0, A \supset f \vdash f$, from which, applying III to the first relation and then I we easily infer $\Gamma_0 \vdash f$ - which is impossible. We shall make use of these simple properties of maximal consistent sets without specific reference.

Returning, now, to our construction, we consider the system $\mathfrak{S}_1$

and build up a set of its w.f.f.'s, Γ_1, containing Γ_0, as follows.

Select the first formula (without free variables) of Γ_0 (in the standard ordering of its formulas) which has the form $(\exists a)A$ for some individual variable a and w.f.f. A (unabbreviated: $(a)(A \supset f) \supset f$.) Add to Γ_0 the result A^* of substituting U_{00} for all free occurrences of a in A. The result will be a consistent set.

For suppose $\Gamma_0, A^* \vdash f$, whence by III, $\Gamma_0 \vdash A^* \supset f$. Since Γ_0 contains neither U_{00} nor any free individual variable we can, in the proof of $A^* \supset f$ (from the "assumptions" of Γ_0) replace U_{00} throughout by the free variable a to obtain a proof $\Gamma_0 \vdash A \supset f$, and by II, $\Gamma_0 \vdash (a)(A \supset f)$. Using 10 and I we then obtain $\Gamma_0 \vdash (\exists a)A \supset f$, which together with I and our assumption $\Gamma_0 \vdash (\exists a)A$ yields $\Gamma_0 \vdash f$, contrary to the consistency of Γ_0.

We continue to choose, one after another in standard order, the formulas of Γ_0 of the form $(\exists a)A$ and add formulas A^* to our set based on Γ_0, each time using a new one of the symbols U_{0i}. When all such formulas have been added we enlarge the resulting set to a maximal consistent set in manner described for Γ_0. This is the set we call "Γ_1".

The process whereby Γ_1 is obtained from Γ_0 may be repeated without change (using the symbols $\{U_{1i}\}$) to obtain a maximal consistent set, Γ_2, of formulas of $\mathcal{S}_2$ containing no free individual variables. And further repetitions lead to such sets Γ_i for each $\mathcal{S}_i$ $(i = 0,1,2,\ldots)$. Finally we let Γ_ω be the union of all of these Γ_i. It is easily verified that Γ_ω is a maximal consistent set of

formulas of $\mathcal{S}_\omega$, and has the simple properties derived above for Γ_0.

We now show that the formulas of Γ_ω, and hence certainly those of Λ, are simultaneously satisfiable in a denumerable domain of individuals. In fact, we take as our model $\mathfrak{J}$ the set of symbols $\{u_{ij}\}$. And we make the following fundamental assignment to the formulas of $\mathcal{S}_\omega$.

To each formula of $\mathcal{S}_\omega$ containing no free occurrence of an individual variable we assign an element of D, T or F, according as the formula is or is not in Γ_ω. If A is a w.f.f. of $\mathcal{S}_\omega$ containing just the free individual variables $a_1, a_2, \ldots, a_n$ we assign that element of C_n whose value, for any individuals $u_{i_1 j_1}, u_{i_2 j_2}, \ldots, u_{i_n j_n}$ as arguments, is T or F according as A^+, the formula obtained from A by replacing free occurrences of a_k $u_{i_k j_k}$ $(k = 1,2,\ldots,n)$, is or is not in Γ_ω. (Note that A^+ has no free individual variables.)

We now show that under this fundamental assignment the basic components of formulas have the same meanings prescribed in the definition of satisfiability (p.3).

(i) Since Γ_ω is consistent it does not contain f; hence under the fundamental assignment f is assigned F.

(ii) If A has been assigned T then $\Gamma_\omega \vdash A$, whence by 1. and I, $\Gamma_\omega \vdash B \supset A$ so that $B \supset A$ is assigned T for each B. If A has been assigned F then $A \notin \Gamma_\omega$ so that $\Gamma_\omega \vdash A \supset f$; whence by 9. and I $\Gamma_\omega \vdash A \supset B$ so that $A \supset B$ is assigned T for each B. Finally suppose that A has been assigned T and B has been assigned F, i.e. $\Gamma_\omega \vdash A$ and $\Gamma_\omega \vdash B \supset f$. Using 7 and I (twice) we get $\Gamma_\omega \vdash A \supset B \supset f$ so that $A \supset B \notin \Gamma_\omega$ and $A \supset B$ is assigned F. In all cases we see

that $\supset$ has the standard meaning under the fundamental assignment.

(iii) Suppose the value T has been assigned to $(a)A$, so that $\Gamma_\omega \vdash (a)A$. Then if u_{ij} is any individual and B is obtained from A by substituting u_{ij} for all free occurrences of a, we see from 5 and I that $\Gamma_\omega \vdash B$, so that A has the value T for all individuals a. On the other hand if $(a)A$ has been assigned F (i.e. $\Gamma_\omega \vdash (a)A \supset f$) we see by 11. and I that $(\exists a)(A \supset f) \in \Gamma_\omega$. But this means that for some i, $(\exists a)(A \supset f) \in \Gamma_i$ whence for some j the result Q of substituting u_{ij} for a in $A \supset f$ is in Γ_{i+1}, hence in Γ_ω. Thus Q has the value T and so A has the value F for at least the one individual u_{ij}. Thus the universal quantifier has its standard meaning under the fundamental assignment.

(iv) Under the convention we have implicitly adopted, that the symbols u_{ij} (as symbols of the formal system $\mathfrak{S}_\omega$) shall denote themselves (as individuals of the domain $\mathfrak{d}$) it is clear that $C(a_1, \ldots, a_n)$ has the standard meaning "C has the value T for the arguments $a_1, \ldots, a_n$" under the fundamental assignment.

Among other provisions, the fundamental assignment associates a truth-value, T or F, with each propositional variable or constant, since these are w.f.f.'s without free individual variables. Further, if C is any functional variable of degree n a unique element of $\mathfrak{C}_n$ is associated with it since the fundamental assignment assigns the same element of $\mathfrak{C}_n$ to each of the w.f.f.'s $C(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ are any distinct individual variables. Thus, since no free individual variables occur in the formulas of Γ_ω the fundamental assignment furnishes a valuation (with respect to $\mathfrak{d}$) for each of

these formulas.

These valuations satisfy the formulas for which they have been furnished. For we have seen that the value of a w.f. formula A, when calculated from the valuation according to the standard meanings of its basic components is the same value as given directly to A by the fundamental assignment - and this value is T for each formula in Γ_ω.

Thus the formulas of Γ_ω, and so of A, are simultaneously satisfiable in the domain whose elements are the symbols $\{u_{ij}\}$, and the proof of theorem I is complete. As we have previously seen, we have the

Corollary I. The pure first-order functional calculus is complete.

Strictly, the completeness of the propositional calculus does not follow as an immediate consequence of Corollary I: we can only infer that the valid formulas of the propositional calculus are all provable within the first order functional calculus. The proof of theorm I would have to be carefully inspected to ascertain that only the axioms and rules of the propositional calculus need enter into the formal proofs of tautologies. Actually, if a separate proof of the completeness of the propositional calculus is desired the whole apparatus of the sequence of Γ_i's may be discarded along with the construction of the domain of u_{ij}'s, and a "fundamental assignment" made directly from Γ_0 (instead of Γ_ω).

Another Corollary which we may infer from Theorem I is the Skolem-Löwenheim theorem. Löwenheim was the first to notice and show

that any w.f.f. of the first order functional calculus which was satisfiable in some domain was also satisfiable in a denumerable domain. Skolem generalized this result to the case of simultaneous satisfiability of a (finite or denumerably infinite) set of formulas. In a paper of 1929 (Skolem [1]) he gives two proofs for this generalization; a simple one using the axiom of choice, and a more delicate one which avoids this tool.

Since a set of formulas (of the pure functional calculus of first order) which are simultaneously satisfiable in some domain cannot be inconsistent (The rules of inference preserve satisfiability with respect to any given domain) we infer at once from Theorem I the Corollary II. If a set of w.f.f.'s are simultaneously satisfiable in some domain, they are also simultaneously satisfiable in a denumerable domain.

Finally we notice that if Λ is an inconsistent set of formulas, the formal proof $\Lambda \vdash f$ involves only a finite number of elements of Λ so that a certain finite subset of Λ cannot be simultaneously satisfiable in any domain. Thus we obtain, as first formulated in Gödel[1],

Corollary III. A set Λ of w.f.f.'s is simultaneously satisfiable if and only if every finite subset is simultaneously satisfiable.

Applied and extended functional Calculi

We may consider altering the basic vocabulary of the pure

functional calculus of first order by adding a finite or effectively enumerable set of propositional, individual, and functional constants, and possibly deleting the propositional and functional variables. In such an applied functional calculus it is usual to add to the rules and axioms given above a fixed set of axioms which are intended to express the properties of the entities denoted by the constants of the system. Well known mathematical axiom systems of geometry and algebra come under this classification.

For such an applied calculus, Theorem I and Corollaries II and III will continue to hold. Indeed, propositional and functional constants are indistinguishable from variables of the same type in the way in which they function in our formulation of the calculi, since we have no substitution rule for variables. Individual variables differ from constants in that they may be bound. When individual constants occur, however, it is only necessary to modify the proof of Theorem I by adding these constants of the system to the a_{ij}'s in forming $\mathfrak{J}$.

The statement that the pure first-order functional calculus is complete may be interpreted as asserting that the intuitive concepts of falsehood, material implication (if...then...), and universal quantification for individuals (all), are adequately characterized by the given rules and axioms of the formal system. In the case of an applied calculus, Theorem I assures the existence of <u>some</u> model for a consistent axiom system, but the constants of the system may have different meanings from those intended a priori in setting up the system.

The question whether formal rules of inference together with a given axiom system adequately characterize a priori meanings intended for various constants of the system cannot, of course, be determined generally. In section IV of this paper we will examine several important special cases. Here, however, we should like to treat the simple example of the relation of equality (between individuals).

Consider, then, the addition to the pure functional calculus of a primitive 2nd-degree functional constant, Q, and signify our intention to have this denote the relation of equality by abbreviating $\alpha = \beta \rightarrow Q(\alpha, \beta)$, for any individual variables α and β, and adding the axioms (for each w.f.f. A):

$\alpha = \alpha$

$\alpha = \beta \supset . A \supset B$, where B is obtained from A by replacing all free occurrences of α by β, providing that no such occurrence of α is in a w.f. part of A of the form $(\iota)C$.

The construction carried out in the proof of theorem I yields a denotation for "Q" which is a certain relation holding between various pairs of the symbols (individuals) U_{ij} - but this is not, in general, the relation of equality. We can, however, modify the construction by noticing that from special cases of the second axiom schema for equality we obtain $\vdash \alpha = \beta \supset \beta = \alpha$ and $\vdash \alpha = \beta \supset . \beta = \gamma \supset \alpha = \gamma$, so that Q denotes an equivalence relation among the U_{ij}'s. We may then form a new domain of individuals, $\mathfrak{J}'$, whose elements are the equivalence classes determined by this relation among the symbols U_{ij}. The valuations with respect to $\mathfrak{J}$ obtained from the fundamental assignment may be transformed into valuations with respect to $\mathfrak{J}'$ in the natural

and obvious way, since the second axiom schema for equality guarantees that formulas A and B are in or out of Γ_ω together whenever they differ only by occurrences of u_{ij} in one at places where u_{kl} appears in the other, providing $u_{ij} = u_{kl}$ is in Γ_ω.

Thus if we add identity to the list of meanings of basic components of w.f.f.'s (p.3) and alter the definitions of validity and satisfiability to include the enlarged list, we may state Theorem I and Corollary I for this applied system.

As a final generalization of the results of this section we consider an extended functional calculus of first order obtained by adding a non-denumerable infinity of primitive variables and/or constants to some pure or applied calculus. Such a construct can no longer be regarded as a logic or a proper lingual system, and the notion of symbol must be made fully abstract. Nevertheless the meaning of such a structure seems clear enough (within the context of classical abstract mathematics,) and the usefulness of studying it will be made patent by the results of the next section.

A first axiomatization of the notion of symbol and the operation of juxtaposition appears in Tarski[1]; a later treatment may be found in Quine[2] in the chapter on Syntax. Both of these accounts needlessly limit consideration to a countable domain of "symbols", but their essential content may be preserved when this restriction is removed. Alternatively the theory of symbolic manipulation may be regarded as a part of the theory of finite sequences, and the operation of juxtaposition defined rather than taken as primitive. We shall not choose

between these alternatives or go into any formal detail here, but will use the intuitive concepts without further justification.

We see at once that the construction and essential steps of the proof of Theorem I can be carried out provided that instead of the symbols u_{ij} we are furnished with a set of symbols $u_{i\alpha}$ $(i = 0,1,2,\ldots)$, where α ranges over a set having the same cardinal number as the set of primitive symbols of the extended functional calculus; and provided further that these symbols can be well-ordered in place of the standard enumeration of formulas upon which we relied in the original proof.

We incorporate these observations in Theorem II. If Λ is any consistent set of w.f.f.'s of an applied, extended, first order functional calculus (which may include identity as a basic component of formulas), then the elements of Λ are simultaneously satisfiable in a domain of individuals having the same cardinal number as the symbols of the formal system, providing these can be well-ordered.

Corollary. A set of formulas Λ of an applied extended calculus is simultaneously satisfiable if and only if each finite subset of Λ is so satisfiable.

PART II

APPLICATIONS TO ALGEBRA

We begin this section with a description of the way in which the elementary portion of ring-theory may be formalized as an applied first-order calculus. For this purpose we drop all propositional and functional variables from the vocabulary of the pure functions-calculus and add two functional constants, "Σ" and "Π", of third degree, and the binary functional constant Q, for identity, as introduced in the previous section. In the intended interpretation for this system $\Sigma(a,b,c)$ is to mean "the sum of a and b is c", and $\Pi(a,b,c)$ means "The product of a and b is c".

As axioms we take schemata 1-5, the two schemata for equality, and the following ring-axioms:

R1. $(a)(b)(\exists c)\,\Sigma(a,b,c)$

R2. $(a)(b)(c)(d).\,\Sigma(a,b,c) \supset .\,\Sigma(a,b,d) \supset c=d$

R3. $(a)(b)(\exists c)\,\Pi(a,b,c)$

R4. $(a)(b)(c)(d).\,\Pi(a,b,c) \supset .\,\Pi(a,b,d) \supset c=d$

These axioms express the fact that Σ and Π are instances of that particular kind of ternary relation which arises from a binary operation. They are required because we have kept our sentence structure simple by not providing directly for symbols denoting operations leading from individuals to individuals $(+, \times)$.

R5. $\Sigma(a,b,d) \supset .\,\Sigma(d,c,e) \supset .\,\Sigma(b,c,f) \supset \Sigma(a,f,e)$

R6. $\Sigma(a,b,c) \supset \Sigma(b,a,c)$

R7. $(a)(b)(\exists c)\,\Sigma(a,c,b)$

Axioms R5-7 are the associative, commutative, and equation-solvability laws for addition.

R8. $\Pi(a,b,d) \supset .\ \Pi(d,c,e) \supset .\ \Pi(b,c,f) \supset \Pi(a,f,e)$

R9. $\Sigma(b,c,d) \supset .\ \Pi(a,d,e) \supset .\ \Pi(a,b,f) \supset .\ \Pi(a,c,g) \supset \Sigma(f,g,e)$

R10. $\Sigma(b,c,d) \supset .\ \Pi(d,a,e) \supset .\ \Pi(b,a,f) \supset .\ \Pi(c,a,g) \supset \Sigma(f,g,e)$.

Axioms R8-10 are the associative law for multiplication and the left and right distributive laws.

When the system is completed by adding Rules of Inference I and II it is adequate for establishing all the elementary theorems which hold in general for rings. Concepts and theorems of a non-elementary nature, however, (i.e. those which require quantification on sets of elements rather than only on the ring elements themselves, such as the existence of ideals of various kinds, etc.) are beyond the range of this system.

We may consider the addition of further primitive constants and axioms containing them in order to study the elementary theory of some particular ring. For example if we were interested in the ring of integers we could add a denumerable set of individual constants, v_n ($n = 0, \pm 1, \pm 2, \ldots$) and all of the following infinite set of axioms:

N1. $\sim(v_n = v_m)$, for $n \neq m$

N2. $\Sigma(v_n, v_m, v_{n+m})$

N3. $\sim\Sigma(v_n, v_m, v_j)$ for $j \neq n+m$

N4. $\Pi(v_n, v_m, v_{n\cdot m})$

N5. $\sim\Pi(v_n, v_m, v_j)$ for $j \neq n \cdot m$

Any model satisfying R1-10 will be a ring in which Σ and Π denote the relations $a+b=c$ and $a\,b=c$ respectively. If, in addition, the axioms N1-5 are satisfied the ring will contain a subring isomorphic to the ordinary integers.

We now illustrate how this mode of formalizing ring theory may be combined with the results of Part I to yield far-from-elementary results. In particular we shall focus our attention on the representation theorem for boolean rings.

A boolean ring is a ring in which each element satisfies the law $a \cdot a = a$ It is easily verified that the set of all subsets of some fixed domain forms a boolean ring under the operations of intersection $(a \cap b)$ as ring multiplication and union (modulo 2), or symmetric difference, $((a \cup b) - (a \cap b))$, as ring addition. We shall give a complete proof of the following Theorem III. Every boolean ring (whose elements can be well-ordered) is isomorphic with some subring of the ring of all subsets of some fixed domain of individuals (in which the ring operations are intersection and union (modulo 2)).

This theorem was first proved in Stone[1] by a method with involves considerable development of ideal theory for boolean rings. Although the present proof is much simpler, it fails to provide the information about ideals in boolean rings which is of interest in itself. On the other hand much of the development of this material can be simplified if a short, independent proof of the representation theorem is available first.

The lemmas with which our proof begins closely parallel theorems

appearing in Chapter I of Stone's paper, but we give them here for the sake of completeness and because they are interspersed with non-essential results in their original context.

Lemma 1. In a boolean ring we have $ab=ba$ and $a+a=0$ for each a, b.

Proof.

We have $(a+a)(a+a)=a+a$ since each element is idempotent. Hence,

using distributivity, $aa+aa+aa+aa=a+a$

by idempotency, $a+a+a+a=a+a$

by cancellation, $a+a=0$

Also $(a+b)(a+b)=a+b$

whence as above $ba+ab=0$

therefore $ba=ba+ba+ab=ab$.

We shall use these rules of calculation without explicit reference.

Definition. An atomic element a, of a boolean ring, is a non-zero element such that $ab=0$ or $ab=a$ for every b. A boolean ring may or may not have atomic elements. If a_1 and a_2 are distinct atomic elements then $a_1a_2=0$; for $a_1a_2\neq 0$ implies both $a_1a_2=a_1$ and $a_1a_2=a_2$, hence $a_1=a_2$.

Lemma 2. In a finite boolean ring we have for each non-zero element b the existence of an atomic element a such that $ab=a$.

Proof.

If b is non-zero but not itself atomic there is a b_1 such that $0\neq bb_1\neq b$. If bb_1 is not atomic we can find a b_2 such that

$0 \neq b b_1, b_2 \neq b b_1$. By the law of idempotency we have $b b_1 \neq b$ implies $b \neq b_1$, and $b b_1, b_2 \neq b b_1$ implies $b_2 \neq b$ $b_2 \neq b_1$. Since we can have at most a finite number of distinct factors, if we continue in this way we must eventually obtain an atomic element $a = b b_1, b_2 \ldots b_n$ for which, clearly, $a b = a$.

Lemma 3. If b is a non-zero element of a finite boolean ring, and $a_1, a_2, \ldots a_n$ are the full set of distinct atomic elements such that $a b = a$, then $b = a_1 + a_2 + \ldots + a_n$.

Proof.

Suppose a is an atomic element distinct from each a_i, so that $a a_i = 0$ and $ab = 0$. Then $a(b + a_1 + a_2 + \ldots + a_n) = 0$. If $a = a_i$ then $a(b + a_1 + \ldots + a_n) = a_i + a_i = 0$. Thus for each atomic element a we have $a(b + a_1 + \ldots + a_n) = 0$ whence by Lemma 2 $b + a_1 + \ldots + a_n = 0$, from which Lemma 3 follows immediately.

Lemma 4. A finite boolean ring is isomorphic to the ring of all subsets of its atomic elements (with the usual set-theoretic operations for ring addition and multiplication).

Proof.

By lemma 3 we have a natural one-one correspondence between the elements of the ring and the subsets of its atomic elements (with the zero element corresponding to the empty set). It is easily verified that the sum of two finite sums of atomic elements is another such sum, whose summands are the union (modulo 2) of the two sets of summands entering the sum; while the product of two such sums is a sum whose summands are the atomic elements in the intersection of the two sets of summands entering the product. This proves the lemma.

Lemma 5. A finite number of elements of a boolean ring generate a finite subring.

Proof.

By the distributive law each element of the generated subring can be expressed as a sum of products of the given elements. By the commutative and idempotent laws for multiplication multiple factors may be reduced to single factors in each term. By the commutative and nilpotent laws for addition no summand may appear twice. Hence lemma 5.

Let "$At(a)$" abbreviate the formula of the calculus which expresses that a is an atomic element:

$$At(a) \rightarrow (\beta)(c).\ \Pi(a,\beta,c) \supset c=a \vee c=0$$

Let "P" abbreviate the formula which asserts that the domain of individuals forms a ring which is isomorphic to a ring of subsets of its atomic elements:

$$a<b \rightarrow \Pi(a,b,a)$$

$$P_1 \rightarrow (a)(\beta)(c).\ \Sigma(a,\beta,c) \equiv (d).\ At(d) \supset [(d<a \vee d<\beta) \sim (d<a \cdot d<\beta)] \equiv d<c$$

$$P_2 \rightarrow (a)(\beta)(c).\ \Pi(a,\beta,c) \equiv (d).\ At(d) \supset [d<a \cdot d<\beta] \equiv d<c$$

$$P \rightarrow P_1 \cdot P_2 \cdot (a).\sim(a=0) \supset (\exists\beta)At(\beta) \cdot \beta<a.$$

Now suppose that $\mathfrak{R}$ is any given infinite boolean ring. We consider an extended calculus involving only the functional constants Σ, Π, Q, a denumerable set of individual variables, and a set of individual

constants, v_n, where "n" ranges through the elements of R. In addition to the axioms of schemata 1-5 and the two schemata for equality we consider the set of formulas, Λ, consisting of R1-10, P, and N1-5 (with "n", "m", and "j" ranging thru' R rather than the ordinary integers).

Now any finite subset of Λ consists of formulas which are simultaneously satisfiable. For such a subset contains only a finite number of symbols v_n. The elements of R corresponding to these symbols (i.e. appearing as their subscripts) generate a finite boolean ring, by lemma 5, which satisfies the given finite subset of axioms N1-5, all of R1-10 (being a ring), and P by lemma 4. Hence by Theorem II and its corollary (and using the hypothesis that the elements of R can be well-ordered) the formulas of Λ are all simultaneously satisfiable (in a domain having the same cardinality as R). But a domain which satisfies all the formulas of Λ must be a ring isomorphic to a ring of subsets (of its own atomic elements), containing the given ring R (isomporphically) as a subring. This proves Theorem III.

The proof of Theorem III falls naturally into two parts: a series of lemmas highly specific to the theory of boolean rings, and a concluding argument obviously capable of wide generalization. We formulate the general situtation in a separate theorem.

Definitions. A type of algebraic structure is defined by a set of axioms involving an undefined set of elements and a specified set of relations,

each relating a finite number of arguments (these relations may arise from operations). Only individual variables may be quantified in the axioms. A particular algebraic structure is a model satisfying such an axiom system. The basic sentences for a particular structure are those which assert for each of the algebraic relations whether or not they hold for each particular finite set of elements (As N1-5). An elementary property of a type of algebraic structure is a property expressed by a formula involving logical symbols, constants for the algebraic relations, and variables ranging only over the individuals (not over sets).

Theorem IV. A necessary and sufficient condition that a given algebraic structure be a sub-structure of an algebraic structure of the same type which has a given elementary property P, is that it be possible to imbed each finite number of elements of the given structure (in a structure of the same type having the property P) in such a way that the basic sentences for these elements are preserved.

The necessity is immediate. The method of proving sufficiency in any particular case is clear from the latter part of the proof of Theorem III. As a further application of Theorem IV we may mention the extension of the representation theorem for boolean rings to the case of distributive lattices, as in Birkhoff[1] (p. 76).

Given a particular description of a particular property of some type of algebraic structure it is generally possible in a mechanical manner to decide whether that description satisfies the condition used

in defining "elementary property". If the decision is affirmative the given property is elementary. But if it is negative, we have no general means of determining whether the property is elementary in virtue of some other description of it which may satisfy the required condition.

As an example, consider the case of algebraic fields, defined by the axioms F consisting of R1-10 together with the two following:

F1. $\Pi(\alpha, \beta, \gamma) \supset \Pi(\beta, \alpha, \gamma)$

F2. $(\alpha)(\beta)(\exists \gamma)\, \Pi(\alpha, \gamma, \beta)$

The property of a field to be of characteristic two may be defined by the condition that the intersection of all subfields be isomorphic to the field of ordinary integers (mod 2). In this form the definition involves variables ranging over certain sets of field elements (the subfields), and so does not suggest that the property in question is an elementary property of fields. On the other hand this is indeed the case, as seen from that fact that a necessary and sufficient condition for a field to be of characteristic two is that for each element a we have $a + a = 0$

We capitalize on this observation in the following Theorem V.

If for an infinite sequence of prime numbers q_i there exist fields of characteristic q_i having a certain elementary property P of fields, then there exists a field of characteristic zero having the property P (and containing at most a denumerable number of elements.).

For proof we have only to consider the set of formulas, Λ, consisting of: the formula expressing the elementary property P, the field axioms F, and the following infinite list of formulas:

$$\sim(a)\, \Sigma(a, a) = 0$$

$$\sim(a)\, \Sigma(a, a, \beta) \supset \Sigma(a, \beta, 0)$$

$$\sim(a)\, \Sigma(a, a, \beta) \supset \Sigma(a, \beta, \gamma) \supset \Sigma(a, \gamma, 0)$$

The formulas of any finite subset of Λ will be simultaneously satisfied by a field of characteristic q_i having the property P, provided the formulas of the above list which are in the subset do not extend beyond the q_i^{th}. By the hypothesis of Theorem V such a field always exists. Hence by Theorem I and its Corollary III we obtain the simultaneous satisfiability of all of the formulas of Λ in some denumerable domain. This model is the one whose existence is asserted in the conclusion of Theorem V.

Corollary. If for an infinite sequence of prime numbers q_i, every field of characteristic q_i has the elementary property P, then any field of characteristic zero can be imbedded (as a subfield) in a field (of the same cardinality) having the property P.

Proof.

We take any given field and form an extended calculus by adding individual constants v_n, n ranging over elements of this field. We then add to the set of formulas Λ mentioned in the proof of Theorem V, all of the basic sentences for the given field (expressed by means of the constants v_n). The corollary is an immediate consequence of the simultaneous satisfiability of this enlarged set of formulas, which can be established by means of the axiom of choice plus Theorem II and its corollary.

The required proof that every finite subset of this set of formulas is satisfiable is quite straightforward provided one has the Lemma. Any finite set of elements taken from any field of characteristic zero can be imbedded in some field of prime characteristic q for all sufficiently

large p (in such a way that all of the basic sentences involving these elements are preserved).

We give only an outline of a proof of this lemma, the idea of which was suggested by E. Artin.

First we can classify our finite set of elements: some will be elements of the prime field (which we can represent as ordinary rational numbers); then there will be a finite sequence of elements each transcendental over the field obtained by adjoining the others to the prime field (these can be represented by letters $x, y, \ldots, z$); and finally the remaining elements will be algebraic over the field $R(x, y, \ldots, z)$ - and these can be represented as algebraic functions in terms of the letters $x, y, \ldots z$.

Next we notice that all of the basic sentences involving these elements can be preserved if we "specialize" the letters to suitably chosen algebraic numbers (and replace the algebraic functions involving the letters, by an appropriate set of their values (assumed when the letters are specialized)). This can be established by using a theorem of Van Der Waerden[1] (§92).

Finally we notice that the ideal generated by the resuling set of algebraic numbers can have at most a finite number of prime ideals as divisors. The quotient field with respect to any other prime ideal satisfies the requirements of the lemma.

This completes our proof of the corollary to Theorem V.

Although Theorem V is principally intended to show the existence of certain fields it may also be used in the classification of properties.

Thus we have.

Corollary II. The property of a field to be of prime characteristic, is not elementary. However, for any particular prime number, p, the property to be of characteristic q is elementary.

The complement of an elementary property is also elementary, for if a sentence expressing the property contains only individual variables within quantifiers its negation will also satisfy this condition. This enables us to infer, for example, that if every field of characteristic zero has a given elementary property, then for at most a finite number of primes p can there be fields of characteristic p failing to have that property (Theorem V); further, that the property to be characteristic zero is not an elementary property (Corollary II).

Although being of characteristic zero is not an elementary condition on fields it can be expressed in a first order calculus by means of an infinite set of formulas, as in the proof of Theorem V. Such a property we can call quasi-elementary, and we then have an example of a property which is quasi-elementary but not elementary. However, the complementary property, to be of finite characteristic, is not even quasi-elementary, for Theorem V can clearly be strengthened to cover quasi-elementary properties without change in proof. In effect, this means that we can handle infinite conjunctions (in a sense) within first order calculi without being able to deal with corresponding infinite disjunctions.

A survey of our applications of the techniques of formal logic to algebra show two basic ideas to be involved.

A. The statement and proof of theorems (even though these are

about infinite aggregates) always involves finite operations performed upon finite sequences of symbols.

B. Nevertheless the formalism for discussing any abstract system has, to a certain extent, the same structure as its subject matter. (This idea has been expressed in many places; e.g. Ducasse[1]. In an extreme form it becomes the thesis of nominalism. It is made precise and capitalized upon by our device of using the set of basic sentences in an extended calculus.)

In the following sections we shall see that these conditions continue to obtain, in part, for mathematical systems requiring higher-order calculi for their formalization, but that difficulties of interpretation arise which sharply curtail the possibility of applying our methods to such systems.

PART III

THE CALCULI OF HIGHER ORDER

The pure functional calculus of second order may be obtained from the first-order calculus by altering the definition of "well-formed formula" to permit quantification on variables of all types (rather than only on individual variables) and adjusting the wording of axiom schemata 4 and 5 and Rule of Inference II so that they apply to formulas in which the quantified variables may be of any kind.

Under the standard interpretation intended for the formulas of the second-order calculus the propositional variables range over the two truth values T and F, and the individual variables range over an arbitrary domain of individuals, just as in the case of the first-order calculus. Functional variables of degree N range over the domain of all truth-valued functions of n individual arguments (or n-ary relations, and sets in the case n = 1, with which these functions may be identified)- domains completely determined as soon as the set of individuals, $\mathcal{I}$, is specified. The appearance of a functional variable within a universal quantifier means "For all functions (of the indicated type)...".

In the second-order calculus we may turn our attention to closed formulas - those containing no free variables of any type, - and thus avoid consideration of valuations. We may verify that all closed formulas which are provable in the formal system have the value truth (i.e. express true propositions) no matter which domain of individuals is selected as the model upon which to base the interpretation. Hence we do not expect that for each closed formula A, either A or $\sim A$ is provable; for some closed w.f.f.'s A will be true or false depending on which model they refer to.

We may, however, as in the case of the first order calculus, ask whether every closed formula which is valid in the sense of having the value T for every standard model, is provable in the formal system. The answer, deep-lying, is negative, and may be inferred from the results of "Gödel"[2]. Not only are the axioms and rules of our formulation of the calculus inadequate, but no extension of the system which retains an effective test of the validity of a proposed formal proof can suffice to yield all valid formulas as formal theorems. This situation is usually interpreted as an incompleteness result for the second-order functional calculus.

Another possibility, however, is open: We may consider a wider class of interpretations for our formalism than the standard models which are intended, provide. As a first attempt we may consider allowing the functional variables of degree N to range over an arbitrary set of truth-valued functions (of n individual arguments), rather than over all of them.

But this plan is quickly seen to be unsatisfactory. For example, if the formula " $F(x)$ " assumes as value some truth-valued function, then under the standard meanings for " $\supset$ " and " f ", the formula " $F(x) \supset f$ " represents the function whose value, for any argument x, is the opposite truth-value to that assumed by the function $F(x)$ for the same argument. For this reason we require the domains which constitute ranges for functional variables to be closed under complementation. Again, we should demand that the various domains be closed under the operation of projection, because along with any formula such

"$F(x,y)$" we may form "$(\exists x)F(x,y)$".

More generally, each basic component which enables us to build up complex w.f.f.'s from simpler ones has associated with it, in view of its intended meaning, some operation on the domains over which function variables may range. An arbitrary domain of individuals and a class of truth-valued function-domains closed under all of these operations may be called a general model (as distinguished from the special case of the standard models).

With reference to a general model an appearance of a functional variable of degree n within a universal quantifier means "For all functions of the domain of functions of n arguments..." and we may redefine a valid (closed) formula as one which has the value T for every general model. Again we notice that each provable formula of the calculus is valid, and raise the question whether every valid formula has a formal proof. This time the answer is affirmative, and may be established by extending the technique of our Part I. We shall, however, give no details, but pass at once to the more general case of the functional calculus of order ω.

Rather than amplify the system presented in Part I, we shall consider a more wieldy formulation of a functional calculus containing all finite types, adopting the system given in Church.[3]

Certain symbol-complexes are used as subscripts on the variables and constants of the system to indicate the type to which these refer. They are described as follows:

ι and o are type-symbols;

if α and β are type-symbols, so is $(\alpha\beta)$

These rules define inductively the complete set of type-symbols.

In the standard interpretation intended for the formal calculus the domain 0 consists of the two truth values, ι is the (arbitrary) domain of individuals, and $(\alpha\beta)$ is the type of all functions ranging over β with values in α.

The primitive symbols of the system are given in the following infinite list, where, as above, it is understood that "α" and "β" may be replaced by any type-symbol.

Variables: u_α v_α w_α x_α y_α z_α u'_α v'_α ...

Constants: $N_{(00)}$ $A_{((00)0)}$ $\Pi_{(0(0\alpha))}$

Improper: λ ( )

Functions of n arguments will be identified, in the interpretation, with functions of one argument whose values are functions of n-1 arguments. Classes and relations are identified with functions of one or several arguments, whose values are in type 0. Thus, $\Pi_{(0(0\alpha))}$ denotes a property of classes of elements of type α; in fact it is the property of being the universal class consisting of all elements of type α (so that "$\Pi_{(0(0\alpha))}$" acts as a kind of universal quantifier). The variables with subscript 'α' will have as their range the entire domain of type α. Parentheses will have their ordinary use as symbols for punctuation, and 'λ' will be used as an abstraction operator.

Certain formulas compounded from the primitive symbols are used to refer to elements of the various type-domains. They will be called 'well-formed formulas', and are described by the following rules, which at the same time serve to indicate to which domains they refer.

A variable or constant alone is a w.f.f., and has the type of its subscript.

If $A_{\alpha\beta}$ and B_β are w.f.f.'s of type $(\alpha\beta)$ and β respectively, then

$(A_{\alpha\beta} B_\beta)$ is a w.f.f. of type α.

If A_α is a w.f.f. of type α, and a_β a variable of type β, then

$(\lambda a_\beta A_\alpha)$ is a w.f.f. of type $(\alpha\beta)$.

As indicated in this description, letters of the early part of the alphabet will be used as syntactical variables; capitals for w.f.f.'s, small letters for variables.

Juxtaposition within parentheses is intended to denote application of function to argument. $(\lambda a_\beta A_\alpha)$ will refer to the function whose value, for any argument, x_β, is denoted by the formula obtained from A_α by replacing each occurrence of a_α by a name of x_β.

In writing both w.f.f.'s and subscripts we will practise the omission of parentheses and their occasional supplantation by dots, the principal rules of restoration being first that the formula should be well-formed, secondly that association is to the left, and thirdly that when a left parenthesis is put back for a dot its mate is to be placed as far to the right as possible. (A detailed statement of our usage can be found in Church[3].)

We now wish to give a precise description of the rules for using well-formed formulas to carry meanings with reference to some standard model of type domains. For this purpose it is again necessary to distinguish between bound and free occurrences of a variable a_α in a w.f.f. A_β, the former being an occurrence in a w.f. part of A_β of the form $\lambda a_\alpha B_\gamma$, all other occurrences being free. A w.f.f. of type α containing no free variables will denote an element of the type domain α. A w.f.f. of type α containing n free variables (n > 0) does not denote anything; however, for each assignment of elements of appropriate type as values for the free variables, there will be an associated element of type α determined by the formula. The function of n variables which embodies this mode of association (or rather the

n! functions, depending on the order in which the n variables are taken,) will be dnoted by the abstracts of the formula. The abstracts of a w.f.f. A_α are the formulas $\lambda a_{\alpha_1} \lambda a_{\alpha_2} \ldots \lambda a_{\alpha_n} A_\alpha$, where $a_{\alpha_1}, \ldots, a_{\alpha_n}$ are all of the variables with free occurrences in A_α in any order; if A_α has no free variables it is its own abstract.

The rules for assigning denotations to the abstracts of a given w.f.f. A_α proceed by induction on the length of A_α:

1). A_α is a single primitive symbol. If A_α is a primitive constant it is its own abstract, and its denotation is given in the following list:

N_{oo} is the operation of negation which yields T or F when applied to F or T respectively. A_{ooo} is the relation of inclusive disjunction between truth values, so that $A_{ooo} x_o y_o$ has the value T except when both x_o and y_o are F.

$\Pi_{o(o\alpha)}$ is the function which has the value F for every argument except that function of type $(o\alpha)$ which takes each element of type α into T, and has the value T for this exceptional argument.

If A_α is a variable, a_α, of type α it has the single abstract $\lambda a_\alpha a_\alpha$, and the denotation of this is the function taking each element of the domain α into itself - i.e. the identity function confined to the domain α.

ii) A_α is $B_{\alpha\beta} C_\beta$. If A_α contains no free variables, then neither does $B_{\alpha\beta}$ or C_β. By induction hypothesis each of the latter therefore denotes some element of appropriate type; we

assign as denotation of A_α the value of the function denoted by $B_{\alpha\beta}$ for the argument denoted by C_β. If A_α has the free variables $a_{\alpha_1}, a_{\alpha_2}, \cdots, a_{\alpha_n}$ (in any order), then the denotation of the abstract $(\lambda a_{\alpha_1} \lambda a_{\alpha_2} \cdots \lambda a_{\alpha_n} A_\alpha)$ is determined by the rule that for any arguments $\chi_{\alpha_1}, \cdots, \chi_{\alpha_n}$ its value is obtained by first applying the denotation of $(\lambda a_{\alpha_{i_1}} \cdots \lambda a_{\alpha_{i_p}} B_{\alpha\beta})$ to the arguments $\chi_{\alpha_{i_1}}, \cdots, \chi_{\alpha_{i_p}}$ to obtain an element $f_{\alpha\beta}$; then applying the denotation of $(\lambda a_{\alpha_{j_1}} \cdots \lambda a_{\alpha_{j_q}} C_\beta)$ to the arguments $\chi_{\alpha_{j_1}}, \cdots, \chi_{\alpha_{j_q}}$ to obtain an element g_β, and finally applying $f_{\alpha\beta}$ to g_β to obtain the required element of type α; where $a_{\alpha_{i_1}}, \cdots, a_{\alpha_{i_p}}$ are the free variables of $B_{\alpha\beta}$ selected in order from $a_{\alpha_1}, \cdots, a_{\alpha_n}$, and $a_{\alpha_{j_1}}, \cdots, a_{\alpha_{j_q}}$ are the free variables of C_β selected in order from $a_{\alpha_1}, \cdots a_{\alpha_n}$.

iii) A_α is $(\lambda a_\beta B_\gamma)$, whence α is $(\gamma\beta)$. If B_γ has no free variables then the denotation of A_α is the function which has the constant value denoted by B_γ (for any argument χ_β). If B_γ contains the free variables $a_{\alpha_1}, \cdots, a_{\alpha_n}$ none of which is a_β, then the denotation of the abstract $(\lambda a_{\alpha_1} \cdots \lambda a_{\alpha_n} A_\alpha)$ is determined by the rule that for any arguments $\chi_{\alpha_1}, \cdots, \chi_{\alpha_n}, \chi_\beta$ it has the same value as the denotation of $(\lambda a_{\alpha_1} \cdots \lambda a_{\alpha_n} B_\gamma)$ for the arguments $\chi_{\alpha_1}, \cdots, \chi_{\alpha_n}$. Finally, if B_γ contains the free variable a_β the abstracts of A_α are included among those of B_γ and so have already been assigned denotations by the induction hypotheses.

It is easily seen from rules i, ii, and iii that a unique denotation is assigned to each w.f.f. containing no free occurrence of a variable.

Furthermore if A_α is a w.f.f. containg the w.f.f. B_β as part, and if C_β is of the same type as B_β and A'_α results from replacing B_β by C_β in A_α, then A'_α is well-formed and its abstracts have the same denotations as those of A_α providing the abstracts of B_β and C_β have the same denotations (in some order).

Formulas of type o (i.e. names of truth-values) will be used as sentences. For any formula A_o without free variables rules i, ii, and iii provide a means of writing down a necessary and sufficient condition for A_o to denote T; and we shall use A_o to express the proposition that this condition does in fact hold (for the model or models referred to). A formula expresses a true or false proposition according as it denotes T or F.

Thus, for example, "$\Pi_{o(oo)}(\lambda x_o\ x_o)$" expresses that the identity function (restricted to the type domain o) always has the value T, which is false; whence "$N_{oo}(\Pi_{o(oo)}(\lambda x_o x_o))$", is true, and so "$\Pi_{o(oo)}(\lambda y_o N_{oo}(\Pi_{o(oo)}(\lambda x_o x_o)))$" denotes T.

Formulas of type σ containing free variables will also be used as sentences expressing the condition that they denote T for any elements (of appropriate type) assigned as values to their free variables. Thus $A_{ooo}\, x_o (N_{oo}\, x_o)$ expresses that any element of type o is either T or F.

The manner in which the several basic concepts of logic may be defined in terms of the primitive notions of our system is indicated by the following list of abbreviations.

$(\sim A_o) \rightarrow N_{oo}\, A_o$

$(A_o \vee B_o) \rightarrow A_{ooo}\, A_o\, B_o$

$(A_o\ B_o) \rightarrow (\sim((\sim A_o) \vee (\sim B_o)))$

$(A_o \supset B_o) \rightarrow ((\sim A_o) \vee B_o)$

$(A_o \equiv B_o) \rightarrow ((A_o \supset B_o)(B_o \supset A_o))$

$(a_\alpha) B_o \rightarrow \Pi_{o(o\alpha)} (\lambda a_\alpha B_o)$

$(\exists a_\alpha) B_o \rightarrow (\sim((a_\alpha)(\sim A_o)))$

$Q_{o\alpha\alpha} \rightarrow \lambda x_\alpha \lambda y_\alpha ((f_{o\alpha})(f_{o\alpha} x_\alpha \supset f_{o\alpha} y_\alpha))$

$(A_\alpha = B_\alpha) \rightarrow Q_{oo\alpha} A_\alpha B_\alpha$

$(A_\alpha \neq B_\alpha) \rightarrow (\sim(A_\alpha = B_\alpha))$

The concepts of the arithmetic of natural numbers may also be defined in terms of the notions of this system, as shown in Church[3], and from these, in turn, the concepts of analysis may be obtained by familiar modes of definition. But we shall not enter into any of these details here.

In order to aid in the recognition of those formulas which express true propositions about every standard model we set up a formal calculus for generating such "theorems" by supplying the following axioms and rules of inference.

Axioms and Axiom schemata.

1. $x_o \vee x_o \supset x_o$
2. $x_o \supset x_o \vee y_o$
3. $x_o \vee y_o \supset y_o \vee x_o$
4. $x_o \supset y_o \supset . z_o \vee x_o \supset z_o \vee y_o$
5. $\Pi_{o(o\alpha)} f_{o\alpha} \supset f_{o\alpha}\, x_\alpha$
6. $(x_\alpha)(y_o \vee f_{o\alpha}\, x_\alpha) \supset y_o \vee \Pi_{o(o\alpha)} f_{o\alpha}$

Rules of Inference.

I. To replace any part A_α of a formula by the result of substituting a_β for b_β throughout A_α, provided that b_β is not a free variable of A_α and a_β does not occur in A_α.

II. To replace any part $(\lambda a_\gamma A_\beta) B_\gamma$ of a w.f.f. by the result of substituting B_γ for a_γ throughout A_β, provided that the bound variables of A_β are distinct both from a_γ and the free variables of B_γ.

III. To infer A_0 from B_0 if B_0 may be inferred from A_0 by a single application of Rule II.

IV. From $A_{0\alpha}\, a_\alpha$ to infer $A_{0\alpha}\, B_\alpha$ if the variable a_α is not free in $A_{0\alpha}$.

V. From $A_0 \supset B_0$ and A_0 to infer B_0

VI. From $A_{0\alpha}\, a_\alpha$ to infer $\Pi_{0(0\alpha)}\, A_{0\alpha}$ provided that the variable a_α is not free in $A_{0\alpha}$.

A <u>formal theorem</u> of the system is a w.f.f. obtained from the formal axioms by a succession of applications of the rules of inference. It may easily be verified that for any standard model each of the axioms expresses a true proposition, and each rule of inference preserves this property. Thus each theorem containing no free variables denotes T with respect to every standard interpretation.

However, because the theory of numbers may be developed within this system the techniques of Gödel[2] apply and may be used to show the existence of a w.f.f. which denotes T for each standard model and yet is not a formal theorem.

We explain this fact by showing that there are other models than the standard ones for which all formal theorems express true propositions. The possibility then arises that the Gödel formulas cannot be proved because they denote F for some non-standard model. This possibility is indeed realized, and contained in our statement of completeness: every formula which denotes T with respect to all models is formally provable.

Let $\mathfrak{D}_0$ be the domain of truth-values T and F, let $\mathfrak{D}_\iota$ be an arbitrary domain of individuals, and let $\mathfrak{D}_{\alpha\beta}$ be an arbitrary set of functions defined on $\mathfrak{D}_\beta$ with values in $\mathfrak{D}_\alpha$. Such a collection of domains we call a **frame**. Given any such frame we may begin assigning elements of its domains as denotations of abstracts of the w.f.f.'s of our formal system according to the rules i, ii, and iii above, except that we replace the phrase "every element of type α" by "every member of the domain $\mathfrak{D}_\alpha$". In general this process of assignment will break down at some stage, for there may be a w.f.f. for which the rules assign as denotation some function defined over $\mathfrak{D}_\beta$ with values in $\mathfrak{D}_\alpha$ which was not included in the set $\mathfrak{D}_{\alpha\beta}$. If no such breakdown occurs then we say that the frame is a (general) model. It is clear that the standard models originally described are included, but the construction of non-standard models is not simple. The reason is that due to the impredicative nature of the definition of "model" one cannot start with

an arbitrary frame and patch it with a simple sequence of additions to the various domains - these latter must be defined simultaneously. We shall, however, describe a method of obtaining all possible models, and show that some are non-standard by exhibiting models each of whose domains is denumerably infinite.

Let Λ be a set of w.f.f.'s such that if these are added to the axioms 1-6[α] a formal proof can be obtained for a certain formula A_0; and any variable a_α which is 'generalized upon' by an application of Rule VI, in the proof, does not occur free in any of the formulas of Λ. In this case we write "$\Lambda \vdash A_0$". If $\Lambda \vdash (x_\alpha)x_0$ we say that Λ is inconsistent, otherwise consistent. Our basic result is given in Theorem VI. If Λ is any consistent set of w.f.f.'s none of which contains a free occurrence of any variable, then there is a denumerable general model such that each element of Λ denotes T with respect to it (in accordance with rules i, ii, and iii above),

We shall make use of the following derived results about the formal calculus which we quote without proof.

VII. The deduction-theorem holds: for any w.f.f.'s A_0 and B_0, and set of formulas Λ, $\Lambda, A_0 \vdash B_0$ implies $\Lambda \vdash A_0 \supset B_0$. (A proof for this system is given in Church[3].)

7. $\vdash A_0 \supset (x_\alpha)x_0 \supset \sim A_0$ implies $\vdash \sim(\exists a_\alpha)A_0$ for any w.f.f. A_0 and variable a_α.

8[α] $\vdash A_0 \supset (x_\alpha)x_0$

9 $A_0, \sim A_0 \vdash (x_\alpha)x_0$

10[α] $\vdash A_\alpha = A_\alpha$

11$^{\alpha}$ $A_\alpha = B_\alpha \vdash B_\alpha = A_\alpha$

12$^{\alpha}$ $A_\alpha = B_\alpha , B_\alpha = C_\alpha \vdash A_\alpha = C_\alpha$

13$^{\alpha}$ $\vdash (\exists x_\alpha) A_\alpha = C_\alpha$ if x_α is not free in A_α

14 $A_0 , A_0 = B_0 \vdash B_0$

15$^{\alpha\beta}$ $B_\beta = B'_\beta \vdash A_{\alpha\beta} B_\beta = A_{\alpha\beta} B'_\beta$

16$^{\alpha\beta}$ $A_{\alpha\beta} = A'_{\alpha\beta} , B_\beta = B'_\beta \vdash A_{\alpha\beta} B_\beta = A'_{\alpha\beta} B'_{\alpha\beta}$

17 $A_0 \vdash A_0 \vee B_0 , \quad A_0 \vdash B_0 \vee A_0$

18$^{\alpha}$ $\Pi_{0(0\alpha)} A_{0\alpha} \vdash A_{0\alpha} B_{0\alpha}$

19$^{\alpha}$ $\sim \Pi_{0(0\alpha)} A_{0\alpha} \vdash (\exists x_\alpha) \sim (A_{0\alpha} x_\alpha)$

20 $\vdash \sim \sim A_0 \supset A_0$

We suppose again that the formulas of the calculus have been given some standard ordering to which we may refer in describing how to choose formulas in various ways. We provide ourselves with a set of constants for each type α: u^{ij}_α $(i,j = 0,1,2,\ldots)$, and calling the original formal system $\mathcal{S}_0$ we form $\mathcal{S}_{j+1}$ by adding the symbols u^{ij}_α $(i = 0,1,2,\ldots)$ as new primitive constants to the system $\mathcal{S}_j$ $(j = 0,1,2,\ldots)$.

Now consider the system $\mathcal{S}_0$ and the given set Λ of its formulas which is consistent and none of whose members contains free variables of any type. We go through the formulas of $\mathcal{S}_0$ one at a time, building a maximal consistent set of formulas without free variables by adding a formula to Λ whenever it is consistent with Λ together with formulas previously added. This maximal set we call Γ_0. If A_0 is a closed formula of type 0 which is not in Γ_0 then $\Gamma_0 , A_0 \vdash (x_0) x_0$ by the maximal property. Hence by VII, $\Gamma_0 \vdash A_0 \supset (x_0) x_0$ and by V and 7, $\Gamma_0 \vdash \sim A_0$ so that $\sim A_0$ is in Γ_0 (since Γ_0 is closed under applications of rules I-VI).

Turning to the system $\mathcal{S}_1$ we build a set, Γ_1, of formulas as follows. Consider the first formula (in the standard ording) of Γ_0 which has the form $(\exists a_\alpha) A_0$ (whatever type α may be). Then add $(\lambda a_\alpha A_0) u^{00}_\alpha$ to Γ_0. The result must be a consistent set of formulas of $\mathcal{S}_1$. For if $\Gamma_0, (\lambda a_\alpha A_0) u^{00}_\alpha \vdash (\chi_0)\chi_0$ then by VII $\Gamma_0 \vdash (\lambda a_\alpha A_0) u^{00}_\alpha \supset (\chi_0)\chi_0$. Since neither u^{00}_α nor a_α (as a free variable) occurs in Γ_0, we may in the proof of $(\lambda a_\alpha A_0) u^{00}_\alpha \supset (\chi_0)\chi_0$ replace the symbol u^{00}_α by a_α throughout, obtaining a proof of $\Gamma_0 \vdash (\lambda a_\alpha A_0) a_\alpha \supset (\chi_0)\chi_0$, whence by II, $\Gamma_0 \vdash A_0 \supset (\chi_0)\chi_0$. Using 8^α we conclude that $\Gamma_0 \vdash \sim(\exists a_\alpha) A_0$ But since $\Gamma_0 \vdash (\exists a_\alpha) A_0$ we see from 9 that $\Gamma_0 \vdash (\chi_0)\chi_0$ which is impossible, since Γ_0 was chosen to be consistent.

We continue through the list of formulas of Γ_0 and when we find the next formula of the form $(\exists b_\beta) B_0$ we add either $(\lambda b_\beta B_0) u^{01}_\alpha$ or $(\lambda b_\beta B_0) u^{00}_\beta$ according as β is α or some other type symbol. Again we retain consistency for our set of formulas. Repeating this procedure through the entire set of formulas Γ_0 we use each of our symbols u^{0j}_α once and have added one formula corresponding to every element of Γ_0 of the form (unabbreviated:) $(N_{00}(\Pi_{0(0\alpha)}(\lambda a_\alpha (N_{00} A_0))))$ We then enlarge this set to a maximal consistent set of closed formulas of $\mathcal{S}_1$ which we call Γ_1 following the procedure used in forming Γ_0.

We move in turn to the systems $\mathcal{S}_2, \mathcal{S}_3, \ldots$ at each stage adding formulas of $\mathcal{S}_{j+1}$ to the set Γ_j to obtain a new set Γ_{j+1} in the same way that Γ_1 was obtained from Γ_0 (except that we use the symbols u^{jk}_α instead of u^{0k}_α. Finally we let Γ_ω be the union of the sets

Γ_j ($j = 0,1,2,\ldots$). It is easily seen that Γ_ω is a maximal consistent set of formulas of the system $\mathcal{S}_\omega$ whose axioms and rules are those of $\mathcal{S}_0$ and whose pririitive symbols consist of all of those of each $\mathcal{S}_j$

From 10, 11 , and 12 we see that the relation which holds between A_α and B_α just when $\Gamma_\omega \vdash A_\alpha = B_\alpha$ is an equivalence relation, and we shall denote by "$(A_\alpha)^*$" The corresponding equivalence class to which A_α belongs. These classes will now be used as a basis in setting up a general model with respect to which each formula of Γ_ω, and so certainly of Λ, denotes T. We start by describing a frame of domains $\mathcal{D}_\alpha$ and making a <u>fundamental assignment</u> of the closed formulas of $\mathcal{S}_\omega$ to the elements of these domains.

$\mathcal{D}_0$ consists of the truth values T and F, and we assign each closed w.f.f. of type 0 in $\mathcal{S}_\omega$ to T or F according as it is or is not in Γ_ω. $\mathcal{D}_\iota$ consists of the classes $(A_\iota)^*$ considered now as individuals, and we assign each w.f.f. A_ι (which is in $\mathcal{S}_\omega$ and contains no free variables) to $(A_\iota)^*$. In the case of the domains $\mathcal{D}_0$ and $\mathcal{D}_\iota$ we verify that two formulas in the same equivalence class are assigned to the same element: for $\mathcal{D}_\iota$ this is obvious from the definition, for $\mathcal{D}_0$ it follows from 14.

Now suppose that $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ have been defined, and that all of the closed formulas of $\mathcal{S}_\omega$ of type α and β have been assigned to their respective elements in such a way that $\Gamma_\omega \vdash A_\alpha = B_\alpha$ implies that A_α and B_α have been assigned to the same element (and similarly in the case of A_β and B_β). We proceed to define $\mathcal{D}_{\alpha\beta}$ and extend the fundamental assignment to formulas of type $(\alpha\beta)$.

Consider any formula $A_{\alpha\beta}$ of $\mathfrak{S}_\omega$ which has the indicated type and in which no variable has a free occurrence. Consider any element x_β of $\mathfrak{D}_\beta$ and let B_β be a formula of $\mathfrak{S}_\omega$ assigned to it by the fundamental assignment. Then the formula $(A_{\alpha\beta}\, B_\beta)$ determines an element y_α of $\mathfrak{D}_\alpha$ to which it has been assigned by our induction hypothesis. By $15^{\alpha\beta}$, if B'_β is any other formula assigned to the same element x_β then $(A_{\alpha\beta} B'_\beta)$ is also assigned to y_α. Thus the formula $A_{\alpha\beta}$ induces a functional correspondence from $\mathfrak{D}_\beta$ to $\mathfrak{D}_\alpha$. We let $\mathfrak{D}_{\alpha\beta}$ be the totality of functions arising from the formulas $A_{\alpha\beta}$ of $\mathfrak{S}_\omega$ in this way and assign any particular formula $A_{\alpha\beta}$ to the function to which it gives rise. By $16^{\alpha\beta}$ if $A_{\alpha\beta}$ and $B_{\alpha\beta}$ belong to the same equivalence class, i.e. if $\Gamma_\omega \vdash A_{\alpha\beta} = B_{\alpha\beta}$, then they are assigned to the same element of $\mathfrak{D}_{\alpha\beta}$.

(In connection with the construction of the domain $\mathfrak{D}_{\alpha\beta}$ it should be noted that the formulas of two different equivalence classes may give rise to functions which agree in value for each argument in $\mathfrak{D}_\beta$. In such a case we must provide distinct elements for the domain $\mathfrak{D}_{\alpha\beta}$, one for each equivalence class, so that we cannot regard functions as determined by their extension. This possibility arises because we have not included a formal axiom of extensionality among the axioms of our calculus, and it may well happen that Γ_ω contains the negation of such a formula thus forcing us into an intensional interpretation. If we wish to limit our consideration to functions in extension we have only to include as axioms the formulas

$$(f_{\alpha\beta})(g_{\alpha\beta})(x_\beta).\ f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta \supset f_{\alpha\beta} = g_{\alpha\beta}.$$

For the system thus enlarged we will have

$$A_{\alpha\beta} \neq B_{\alpha\beta} \vdash (\exists a_\beta).\ A_{\alpha\beta} a_\beta \neq B_{\alpha\beta} a_\beta$$

. Hence if $\Gamma_\omega \vdash A_{\alpha\beta} \neq B_{\alpha\beta}$ and $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are both in $\mathfrak{S}_j$ then for

some k, $\Gamma_{j+1} \vdash A_{\alpha\beta} u^{jk}_{\beta} \neq B_{\alpha\beta} u^{jk}_{\beta}$ so that $A_{\alpha\beta} u^{jk}_{\beta} = B_{\alpha\beta} u^{jk}_{\beta}$ is not in Γ_ω and $A_{\alpha\beta}$ and $B_{\alpha\beta}$ give rise to different functions-in-extension.)

We now complete the proof of Theorem VI by showing that the frame of domains $\mathfrak{D}_\alpha$ constitutes a model for which each formula of $\mathcal{S}_\omega$ denotes the element to which it corresponds under the fundamental assignment. If A_α is a w.f.f. of $\mathcal{S}_\omega$ without free variables we will denote by A^+_α the element of $\mathfrak{D}_\alpha$ with which it is associated under the fundamental assignment. We have in all cases: $A^+_{\alpha\beta} B^+_\beta = (A_{\alpha\beta} B_\beta)^+$. It should be noted, with respect to this equation, that the juxtaposition occurring on the left side of the sign '=' signifies application of function to argument, while the occurrence in the right member denotes the operation of juxtaposing the formulas $A_{\alpha\beta}$ and B_β.

i) Suppose A_α is a single primitive symbol.

If B^+_0 is T so that $\Gamma_\omega \vdash B_0$ then $\sim B_0$ is not in Γ_ω since Γ_ω is consistent. Hence $(N_{00} B_0)^+$ is F. If B^+_0 is F then $\Gamma_\omega, B_0 \vdash (x_0) x_0$ since Γ_ω is maximal consistent; therefore $\Gamma_\omega \vdash B_0 \supset (x_0) x_0$ by VII and $\Gamma_\omega \vdash \sim B_0$ by 7 and V, whence $(N_{00} B_0)^+$ is T. Thus $N^+_{00} T = F$ and $N^+_{00} F = T$ and N^+_{00} is the denotation of N_{00} under the original rule i)...

If B^+_0 or C^+_0 is T so that either $\Gamma_\omega \vdash B_0$ or $\Gamma_\omega \vdash C_0$, then by 17 $\Gamma_\omega \vdash A_{000} B_0 C_0$ so that $A^+_{000} B^+_0 C^+_0$ is T. If neither B_0 nor C_0 is in Γ_ω then $\Gamma_\omega \vdash \sim B_0$ and $\Gamma_\omega \vdash \sim C_0$ whence by 18 $\Gamma_\omega \vdash \sim(B_0 \vee C_0)$ and $B_0 \vee C_0$ is not in Γ_ω. That is, $B^+_0 = F$ and $C^+_0 = F$ imply $(A^+_{000} B^+_0 C^+_0) = F$. Thus A^+_{000} is the denotation of A_{000} according to the original rule i)...

Suppose $\Pi^{\tau}_{0(0\alpha)} A^+_{0\alpha}$ is T, so that $\Gamma_\omega \vdash \Pi_{0(0\alpha)} A_{0\alpha}$

Then by 18^{α} for any B_α $\Gamma_\omega \vdash A_{0\alpha} B_\alpha$ so that for any element B_α^+ of $\mathfrak{D}_\alpha$ we have $(A_{0\alpha} B_\alpha)^+ = A_{0\alpha}^+ B_\alpha^+ = T$. On the other hand if $\Pi_{0(0\alpha)}^+ A_{0\alpha}^+$ is F we have $\Gamma_\omega \vdash \sim \Pi_{0(0\alpha)} A_{0\alpha}$ whence by 19^{α} $\Gamma_\omega \vdash (\exists x_\alpha) \sim (A_{0\alpha} x_\alpha)$. Therefore for some j, $\Gamma_j \vdash (\exists x_\alpha) \sim (A_{0\alpha} x_\alpha)$ and so there is a K such that $\Gamma_{j+1} \vdash (\lambda x_\alpha \sim (A_{0\alpha} x_\alpha)) U_\alpha^{jk}$ and by II $\Gamma_\omega \vdash \sim A_{0\alpha} U_\alpha^{jk}$. Hence $A_{0\alpha} U_\alpha^{jk}$ is not in Γ_ω and $A_{0\alpha}^+$ has the value F for at least the one argument $(U_\alpha^{jk})^+$. So also for the constants $\Pi_{0(0\alpha)}$ we have $\Pi_{0(0\alpha)}^+$ as the denotation of $\Pi_{0(0\alpha)}$ in accordance with the original rule i).

Finally if A_α is a variable, a_α, we consider its abstract $\lambda a_\alpha a_\alpha$ and notice that for any element B_α^+ of $\mathfrak{D}_\alpha$ we have $(\lambda a_\alpha a_\alpha)^+ B_\alpha^+ = ((\lambda a_\alpha a_\alpha) B_\alpha)^+ = B_\alpha^+$ by II so that $(\lambda a_\alpha a_\alpha)^+$ is indeed the identity function as required (by the original rule i) to qualify as the denotation of $\lambda a_\alpha a_\alpha$.

ii). If A_α is $(B_{\alpha\beta} C_\beta)$. contains no free variables then we have at once $(B_{\alpha\beta} C_\beta)^+ = B_{\alpha\beta}^+ C_\beta^+$. By induction hypothesis $B_{\alpha\beta}^+$ and C_β^+ are the denotations respectively of $B_{\alpha\beta}$ and C_β; hence we conclude that $B_{\alpha\beta} C_\beta$ denotes $(B_{\alpha\beta} C_\beta)^+$ by the original rule ii.

Suppose that $a_{\alpha_1}, \ldots, a_{\alpha_n}$ are the free variables of A_α in any given order, and that $a_{\alpha_{i_1}}, \ldots, a_{\alpha_{i_p}}$ are the free variables of $B_{\alpha\beta}$ selected in order from $a_{\alpha_1}, \ldots, a_{\alpha_n}$, while $a_{\alpha_{j_1}}, \ldots, a_{\alpha_{j_q}}$ are the free variables of C_β selected in order from $a_{\alpha_1}, \ldots, a_{\alpha_n}$. By induction hypothesis the denotation of the abstract of $B_{\alpha\beta}$ is $(\lambda a_{\alpha_{i_1}} \ldots \lambda a_{\alpha_{i_p}} B_{\alpha\beta})^+$

and that of C_β is $(\lambda a_{\alpha_{j_1}} \ldots \lambda a_{\alpha_{j_g}} C_\beta)^+$. Hence by ii) of the definition of 'denotation', for any arguments $D_{\alpha_1}^+, \ldots, D_{\alpha_n}^+$ the denotation of A_α is

$$((\lambda a_{\alpha_{i_1}} \ldots \lambda a_{\alpha_{i_p}} B_{\alpha\beta})^+ D_{\alpha_{i_1}}^+ \ldots D_{\alpha_{i_p}}^+)((\lambda a_{\alpha_{j_1}} \ldots \lambda a_{\alpha_{j_g}} C_\beta)^+ D_{\alpha_{j_1}}^+ \ldots D_{\alpha_{j_g}}^+)]^+.$$

$$= [((\lambda a_{\alpha_{i_1}} \ldots \lambda a_{\alpha_{i_p}} B_{\alpha\beta}) D_{\alpha_{i_1}} \ldots D_{\alpha_{i_p}})((\lambda a_{\alpha_{j_1}} \ldots \lambda a_{\alpha_{j_g}} C_\beta) D_{\alpha_{j_1}} \ldots D_{\alpha_{j_g}})]^+$$

$$= [(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} (B_{\alpha\beta} C_\beta)) D_{\alpha_1} \ldots D_{\alpha_n}]^+ \text{ (by rules of inference I, II, III)}$$

$$= (\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} A_\alpha)^+ D_{\alpha_1}^+ \ldots D_{\alpha_n}^+.$$

Thus the denotation of the abstract of A_α is $(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} A_\alpha)^+$.

iii). A_α is $(\lambda a_\beta B_\gamma)$, whence α is $(\gamma\beta)$. If B_γ has no free variables then its denotation is B_γ^+. Since for any argument C_β^+ of D_β we have $(\lambda a_\beta B_\gamma)^+ C_\beta^+ = ((\lambda a_\beta B_\gamma) C_\beta)^+ = B_\gamma^+$ (by Rule of inference II) we see that $(\lambda a_\beta B_\gamma)^+$ is the denotation of A_α. (Our use of Rule of inference II is to assure us (from 10^α) that $\vdash (\lambda a_\beta B_\gamma) C_\beta = B\gamma$ when B_γ contains no free variables. Hence $\Gamma\omega \vdash (\lambda a_\beta B\gamma) C_\beta = B\gamma$ and so $((\lambda a_\beta B_\gamma) C_\beta)^+ = B_\gamma^+$).

Similarly, if B_γ has the free variables $a_{\alpha_1}, \ldots a_{\alpha_n}$ none of which is a_β then our induction hypothesis asserts that the denotation of the abstract of B_γ is $(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} B_\gamma)^+$, and since for any arguments $C_{\alpha_1}^+, \ldots, C_{\alpha_n}^+, C_\beta^+$

we have $(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} (\lambda a_\beta B_\gamma))^+ C_{\alpha_1}^+ \ldots C_{\alpha_n}^+ C_\beta^+$

$= [(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} \lambda a_\beta B_\gamma) C_{\alpha_1} \ldots C_{\alpha_n} C_\beta]^+$

$= [(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} B_\gamma) C_{\alpha_1} \ldots C_{\alpha_n}]^+$

$= (\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} B_\gamma)^+ C_{\alpha_1}^+ \ldots C_{\alpha_n}^+$

we see that $(\lambda a_{\alpha_1} \ldots \lambda a_{\alpha_n} A_\alpha)^+$ is indeed the denotation of the abstract of A as defined under iii).

Finally if B_γ contains the free variable a_β then any abstract A'_δ of A_α is an abstract of B_γ and so has the denotation $(A'_\delta)^+$ by induction hypothesis.

We have shown that every w.f.f. without free variables has its denotation (in accordance with the defining rules i. ii, iii) in the appropriate domain $\mathfrak{D}_\alpha$; hence this frame of domains constitutes a general model. Obviously there are only a denumerable number of elements in each domain, because they are in one-one correspondence with certain equivalence-classes of formulas which form themselves a denumerable set. Finally, since the denotation of any formula A_α is the element A_α^+ assigned to it under the fundamental assignment, every element of Γ_ω, and hence of Λ, denotes T. This completes the proof of Theorem VI.

In analogy to Theorem I of the first section we may infer several corollaries to Theorem VI.

Corollary I. Our formulation of the functional calculus of order ω

is complete.

For suppose A_0 is a closed formula which denotes T with respect to every general model. Then $\sim A_0$ does not denote T for any model and so by Theorem VI itself forms an inconsistent set: $\sim A_0 \vdash (x_0)x_0$. Using VII, 7 and V we infer $\vdash \sim\sim A_0$, and hence by V and 20, $\vdash A_0$. Thus each formula which denotes T for every general model is provable. It follows that any formula which is not provable can be proved independent by exhibiting a denumerable general model with respect to which it expresses a false proposition.

Corollary II. If each of a set of formulas denotes T with respect to some model then the same is true with respect to some denumerable model.

This follows from Theorem VI since the property of denoting T with respect to any given model is preserved by the rules of inference, and hence no set of formulas each expressing a true proposition about a given model can be inconsistent.

Corollary III. If Λ is a set of formulas such that for each finite subset there exists a general model with respect to which the formulas of the subset denote T, then there is a (denumerable) model for which every formula of Λ denotes T.

For if Λ were an inconsistent set there would be a proof of $(x_0)x_0$ involving only a finite subset of its formulas.

Again paralleling the development of Section I we can generalize Theorem VI in two directions. In the first place we may add to our

calculus new primitive symbols in the form of a finite or denumerable set of constants of any type. Then the proof of Theorem VI can be carried through without any change and will provide suitable denotations for the various constants (with respect to the constructed model of domains $\mathfrak{D}_\alpha$) satisfying any consistent axioms involving these constants. In the second place we may add a non-denumerable number of "symbols" to form an extended "calculus", and again the proof used for Theorem VI may be put through provided that the symbols of the system can be well-ordered as a substitute for the standard ordering of formulas used in constructing maximal consistent sets. Thus we have:

Theorem VII. For any applied, extended calculus obtained by adding an arbitrary set of primitive constants to the pure functional calculus of order ω described above, we can find for any set of consistent formulas some general model, and denotations with respect to this model for all of the added constants, such that each formula of the given set denotes T with respect to the model; provided that the symbols of the system can be well ordered. Furthermore the model may be chosen so as to have a cardinal number not greater than that of the set of symbols of the formal system.

As an example of an applied calculus we may add the constants $\iota_{\alpha(o\alpha)}$ and the axioms 9^a of Church[3]:

$$f_{o\alpha} x_\alpha \supset .(\exists y_\alpha)[f_{o\alpha} y_\alpha \supset x_\alpha = y_\alpha] \supset f_{o\alpha}(\iota_{\alpha(o\alpha)} f_{o\alpha}).$$

Any general model for which these formulas denote T will contain description operators which for any class containing just one member have

as value that member.

Alternatively we may use the constants $\iota_{\alpha(o\alpha)}$ to-gether with the axiom(s) of choice:

$$f_{o\alpha}\, x_\alpha \supset f_{o\alpha}\,(\iota_{\alpha(o\alpha)}\, f_{o\alpha}).$$

Models for which these formulas express true propositions will contain choice functions which select one member from each non-empty class of elements of $\mathfrak{D}_\alpha$ (for each α).

We may note that for a system which includes this formulation of the axiom of choice some of the detail in the proof of Theorem VI may be considerably simplified. In fact the constants U_α^{ij} may be dispensed with entirely, and the sequence of systems $\mathfrak{S}_j$ replaced simply by $\mathfrak{S}_o$, the "fundamental assignment" being made with respect to Γ_o instead of Γ_ω. For the function of the U_α^{ij}'s in the proof was to provide some formula U_α such that $\vdash (\lambda x_\alpha A_o) U_\alpha$ whenever $\vdash (\exists x_\alpha) A_o$, and in the presence of the axioms of choice U_α may be taken as $\iota_{\alpha(o\alpha)}(\lambda x_\alpha A_o)$.

Corollary III of Theorem VI may be generalized to Theorem VII:

Corollary. For an extended, applied calculus a model can be found for which each of a given set of formulas denotes T, provided this can be done for each of its finite subsets.

PART IV

APPLIED SYSTEMS OF LOGIC

Having paralleled Theorems I and II for the functional calculus of first order by Theorems VI and VII for the functional calculus of order w, it is natural to attempt to extend the techniques of Part II to obtain applications to types of mathematical structure of such complexity that they can be described only by functional calculi of higher than first order. However, because the consistency of a set of formulas only entails satisfiability with respect to some general model (for which the universal quantifiers of higher type need not mean "really all" but only "all relative to some subset of elements of that type") such applications generally fail.

For example, consider the Theory of topological spaces. If we identify the points of an arbitrary space with the individuals of our domain of type ι, we may formalize this theory by adding to the system of the preceding section (including the axioms of extensionality and choice) a primitive constant " $O_{o(o\iota)}$ " to denote the property (of a point set) of being open, and the following axioms:

T1. $\Pi_{o(o\iota)} X_{o\iota} \supset O_{o(o\iota)} X_{o\iota}$

T2. $O_{o(o\iota)}(\lambda X_\iota . (X_o)X_o)$

T3. $O_{o(o\iota)} X_{o\iota} \supset . O_{o(o\iota)} Y_{o\iota} \supset . O_{o(o\iota)}[\iota_{o\iota(o(o\iota))}(\lambda Z_{o\iota} . (X_\iota)Z_{o\iota} X_\iota \equiv (X_{o\iota} X_\iota . Y_{o\iota} X_\iota))]$

T4. $f_{o(o\iota)} X_{o\iota} \supset O_{o(o\iota)} X_{o\iota} \supset . O_{o(o\iota)}[\iota_{o\iota(o(o\iota))}(\lambda Z_{o\iota} (X_\iota) .$
$Z_{o\iota} X_\iota \equiv (\exists Y_{o\iota}) . f_{o(o\iota)} Y_{o\iota} . Y_{o\iota} X_\iota)]$

The domain of individuals of any standard model which satisfies these axioms (i.e. for which there is a suitable denotation for " $O_{o(o\iota)}$ " such that T1-4 express true propositions) will be a topological space

whose open sets are those X_{α} such that " $O_{0(\alpha)} X_{\alpha}$ " denotes T. However, a general model satisfying T1-4 need not be such a space at all. Hence we cannot expect any general topological imbedding Theorem resembling Theorem IV of Part II.

In some cases, however, we require a functional calculus of higher than first order without needing quantification of variables for sets or functions. An example of an application illustrating this case is the following theorem by C. J. Everett and George Whaples which we quote from abstract 170, Bulletin of the American Mathematical Society, vol. 53 (1947)

Theorem VIII. An arbitrary class of finite sets T_{ξ} possesses a choice function $f(T_{\xi}) = t_{\xi} \in T_{\xi}$, with $f(T_{\xi_1}) \neq f(T_{\xi_2})$ for $\xi_1 \neq \xi_2$, if and only if the union of every k of the T_{ξ} $(k = 1,2,3,\ldots)$ contains at least k distinct elements.

Necessity of the condition is clear. The statement of sufficiency generalizes a well known result of P. Hall for a finite class of finite sets. We shall assume Hall's result and show how our Theorem VII may be used to obtain the generalization immediately.

Let K be any given class of sets T_{ξ} and let " X_{ν} " range through the elements in the union of these sets. We form an extended formal system from the calculus of the preceding section by adding primitive constants $\mathcal{U}_{0\alpha}^{\xi}$ and $\mathcal{U}_{i}^{\nu}$ to provide a name for each element of K as well as for each of its elements. (It may be that T_{ξ_1} and T_{ξ_2} have the same members for some $\xi_1 \neq \xi_2$; in that case we supply two constants $\mathcal{U}_{0\alpha}^{\xi_1}$ and $\mathcal{U}_{0\alpha}^{\xi_2}$). We also add a constant $F_{i(0\alpha)}$.

Now we form a set of w.f.f.'s, Λ, consisting of axioms 1-6";

the axioms of choice for type $\iota(o\iota)$, and the following basic sentences:

$$u^{\xi}_{o\iota}\, u^{\nu_1}_{\iota} \cdot u^{\xi}_{o\iota}\, u^{\nu_2}_{\iota} \cdot \ldots \cdot u^{\xi}_{o\iota}\, u^{\nu_h}_{\iota} \cdot (y_\iota) \cdot u^{\xi}_{o\iota}\, y_\iota \supset \cdot\, y_\iota = u^{\nu_1}_{\iota} \vee y_\iota = u^{\nu_2}_{\iota} \vee \ldots \cdot y_\iota = u^{\nu_h}_{\iota}$$

for each ξ, where $x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_h}$ are the members of T_ξ.

$$F_{\iota(o\iota)}\, u^{\xi_1}_{o\iota} \neq F_{\iota(o\iota)}\, u^{\xi_2}_{o\iota}$$ for each distinct pair ξ_1, ξ_2

And

$$(y_{o\iota})(z_\iota) \cdot y_{o\iota} z_\iota \supset y_{o\iota}(F_{\iota(o\iota)}\, y_{o\iota}).$$

Suppose now that K satisfies the condition that the union of every k of its members ($k = 1,2,3,\ldots$) contains at least k distinct elements. Then by Hall's Theorem there is a model for each finite subset of Λ with respect to which every formula of the subset denotes T. Hence by the corollary to Theorem VII we infer the existence of a single model for which every formula of Λ expresses a true proposition. But this is Theorem VIII.

Set Theory.

The existence of general models for the system of notation on which our calculus is based may be interpreted as proving the possibility of giving various meanings to the word "function" (and its derivative "set") as these are employed in mathematical discourse. When suitable definitions are introduced via abbreviational conventions it may be shown that essentially all the concepts of classical analysis and number theory can be expressed through the formalism we have been considering; while the standard proofs of theorems involving these concepts can all be formalized and subsumed under our

axioms and rules of inference.

The formulation and communication of new concepts and theorems is carried on by means of language in which phrases such as "all sets" (of some specified domain of elements) are interpreted as referring to standard models. An awareness and study of non-standard models may be expected to aid in the intuitive process of discovering theorems and to provide a method for showing the impossibility of proving theorems which may appear true when considered as referring only to standard models.

Skolem [1,2] was the first to point out that the meaning of most mathematical concepts depended on the meaning of "set", and that this concept could never be fully characterized and so always retained a measure of ambiguity within any formal deductive system. His discussion of the relative meaning of "set" is based on a formulation of set theory as an applied functional calculus of first order following the pattern of Zermelo and Fraenkel.

By substituting a single binary functional constant, $\in$, (to denote the relation of membership) for the various functional variables of the pure first order calculus a system for set theory is obtained in which the domain of individuals is taken as the extension of "set". In the standard interpretation there will, of course, be a non-denumerable infinity of sets; but by the Skolem Löwenheim theorem (our corollary II to Theorem I) Skolem shows that no matter what axioms are taken an alternate interpretation of the symbolism is possible in which there are only a denumerable number of sets.

Using our corollary II to Theorem VI we can strengthen this result to the assertion that the notion of "set" cannot be fully characterized by the stronger system of set theory obtained by adding a constant $\in_{o(\iota)}$ (together with the usual axioms involving it) to the functional calculus of order ω. (Such a system does not seem to have been investigated in the literature. The idea of considering it was suggested independently by Prof. Church.)

In connection with such a system the question arises how to interpret domains of higher type since the domain of individuals is supposed to contain 'all sets' (and so functions as special cases). One answer is suggested by the systems of Bernays and Gödel, which distinguish between "sets" and "classes". We can generalize this treatment by postulating the existence of an infinite number of different kinds of aggregates, "sets", "classes", "families", etc., reserving the domain of individuals of the functional calculus of order ω for the sets, and using the domains of higher type for the other kinds of aggregates.

What intuitive picture are we to give to this formidable array of abstract concepts? Perhaps something like this. The very simple intuitive notion of aggregate leads to paradoxes such as those of Russell and Burali-Forti. We discover that these can be avoided (at least in their obvious formulations) by excluding those aggregates which are "too large". We therefore limit our consideration to some fixed collection of aggregates which we call "sets". Once the domain of sets is fixed we can, however, perform simple intuitive operations upon this collection to form new aggregates. For instance we may

consider the collection of all sub-aggregates of the domain of sets. Of course to permit "all" intuitive operations on this domain would lead back to the antinomies, but a few specified operations appear to constitute a safe procedure. From "sets" - a well-defined, proper, sub-domain of our "naive" supply of aggregates, we are thus led to a larger domain which we may call "classes". (Whether we identify a set and the class to which it gives rise in passing to the larger domain, as in Gödel[3], or continue to distinguish them as in Bernays[1], seems to be an unimportant detail.) In the same way we may add further aggregates to our "classes" to form "families" by admitting certain specified operations upon classes. And so on in an indefinite, transfinite sequence.

This process of continuing to build more inclusive domains of aggregates resembles the process of building richer and richer languages from some initial "object language" by a fixed operation such as the addition to any language of terminology expressing the concept of "true sentence" for that language. Indeed these two processes are intimately related. For if we formalize the hierarchy of aggregate-domains within a theory of types, as suggested above, we can apply the ideas of Tarski[1] directly to develop the theory of truth for the language of set-theory within the language of class-theory, and similarly for the higher levels.

Number Theory

As remarked before, the concepts of elementary number theory may be introduced into the pure functional calculus of order ω by definition, a form particularly suited to the present formulation being given in Church[3]. Under this approach the natural numbers are identified with

certain functions. Alternatively we may choose to identify the natural numbers with the individuals making up the domain of type ι. In such a system it is convenient to construct an applied calculus by introducing the constants 0_ι and $S_{\iota\iota}$ and adding the following formal equivalents of Peano's postulates.

P1. $\sim S_{\iota\iota}\, x_\iota = 0_\iota$

P2. $S_{\iota\iota}\, x_\iota = S_{\iota\iota}\, y_\iota \supset x_\iota = y_\iota$

P3. $f_{o\iota}\, 0_\iota \supset .\ (x_\iota)[f_{o\iota}\, x_\iota \supset f_{o\iota}(S_{\iota\iota}\, x_\iota)] \supset f_{o\iota}\, x_\iota$.

The Peano axioms are generally thought to characterize the number-sequence fully in the sense that they form a categorical axiom set any two models for which are isomorphic. As Skolem[1] points out, however, this condition obtains only if "set" - as it appears in the axiom of complete induction (our P3) - is interpreted with its standard meaning. Since, however, we have already seen that any axiometization of the set concept admits non-standard interpretations it follows that we may expect non-standard models for the Peano axioms.

This argument may be somewhat clearer if we consider in detail the usual proof of the categoricity of Peano's postulates. One easily shows that any model for the axioms must contain a sequence of the order-type of the natural numbers by considering the individuals 0_ι, $S_{\iota\iota}\, 0_\iota$, $S_{\iota\iota}(S_{\iota\iota}\, 0_\iota)$, ... and using P1 and P2 to show them distinct and without other predecessors. Then the proof continues as follows. Suppose that the domain of individuals contained elements other than those of this sequence (which we may as well identify with the natural numbers themselves). Then consider the class of individuals consisting

of just the natural numbers. Since it contains zero (0_ι) and is closed under the successor function $(S_{\iota\iota})$ we infer from the axiom of complete induction (P3) that it contains all individuals, contrary to the hypothesis that some individuals were not numbers.

By examining this proof we conclude only that a necessary condition for a general model to satisfy Peano's axioms and at the same time possess a domain of individuals not isomorphic to the natural numbers is that the domain of sets of individuals (type (0ι)) shall not contain the set consisting of just those individuals which are numbers.

Although Skolem[1] indicates that the meaning of "natural number" is relative to the variable meaning of "set" he does not give any example of a non-standard number system satisfying Peano's axioms. In two later papers (Skolem[3,4]), however, he proves that it is impossible to characterize the natural number sequence by any denumerable system of axioms formulated within the first order functional calculus (to which may be added any set of functional constants denoting numerical functions and relations), the individual variables ranging over the "numbers" themselves. For this problem the simple argument used in showing the impossibility of characterizing the set-concept (namely that the axioms will always admit a denumerable model) is, of course, not adequate. However Skolem makes ingenious use of a theorem on sequences of functions (which he had previously proved) to construct, for each set of axioms for the number sequence (of the type described above) a set of numerical functions which satisfy the axioms but have a different order type than the natural numbers.

Using the methods of section III, however, it becomes a simple

matter to construct a model containing a non-standard number system which will satisfy the Peano postulates as well as any preassigned set of further axioms (which may include constants for special functions as well as constants and variables of higher type). We have only to adjoin a new primitive constant a_ι and add to the given set of axioms the infinite list of formulas $a_\iota \neq 0_\iota$, $a_\iota \neq S_{\iota\iota} 0_\iota$, $a_\iota \neq S_{\iota\iota}(S_{\iota\iota} 0_\iota)$, ... Since any finite subset of the enlarged system of formulas is satisfiable it follows from corollary III, Theorem VI that some denumerable model satisfies the full set of formulas, and such a model has the properties sought. By passing to an extended functional calculus and adding a non-denumerable number of primitive constants v_ι^{ξ} together with all formulas $v_\iota^{\xi_1} \neq v_\iota^{\xi_2}$ for $\xi_1 \neq \xi_2$ we may even build models for which the Peano axioms are valid and which contain a number system having any given cardinal.

These same remarks suffice to show more generally that no mathematical axiom system can be genuinely categorical (determine its models to within isomorphism) unless it constrains its domain of elements to have some definite finite cardinal number, providing the logical notions of set and function are axiomatized along with the specific mathematical notions.

The existence of non-standard models satisfying axiom-systems for number theory throws new light on the phenomenon of ω-inconsistency, first investigated by Tarski and Gödel. A formal system is ω-inconsistent if for some formula $A_{o\iota}$ the formulas $A_{o\iota} 0_\iota$, $A_{o\iota}(S_{\iota\iota} 0_\iota)$, $A_{o\iota}(S_{\iota\iota}(S_{\iota\iota} 0_\iota))$ ..... $\sim(\chi_\iota) A_{o\iota} \chi_\iota$ are all provable. Tarski, and later Gödel, showed the existence of consistent systems which were ω-inconsistent. Such a situation merely means that the system must be

interpreted as referring to a non-standard number system whose elements include the natural numbers as a proper subset.

It is generally recognized that all theorems of number theory in the literature can be formalized and proved within the functional calculus of order w with axioms P1-P3 added. (In fact much weaker systems suffice). On the one hand it follows that these theorems can be reinterpreted as true assertions about a great variety of number-systems other than the natural numbers. On the other hand it follows from the results of "Godel[2] that there are true theorems about the natural numbers which cannot be proved by extant methods (consistency assumed).

Now "Godel's proof furnishes certain special formulas which are shown to be true but unprovable, but there is no general method indicated for establishing that a given theorem cannot be proven from given axioms. As remarked in connection with Corollary I to Theorem VI such a method is supplied by the procedure of constructing non-standard models for number theory in which "set" and "function" are reinterpreted. It therefore becomes of practical interest to number-theorists to study the structure of such models.

A detailed investigation of these numerical structures is beyond the scope of the present paper. We may, however, obtain a hint as to their degree of complexity through the following observations. Suppose that a certain model satisfying the Peano postulates contains an element u in addition to the natural numbers $0,1,2,\ldots$. Then since each element has a successor and every non-zero element has a predecessor there will coexist with u elements $\ldots u-1, u, u+1, u+2, \ldots$ in the model. None of these can equal a natural number (for if $u+n=m$) then

$u = m - n$ and they are all distinct. Thus the model consists of individuals having the order type $\omega + \eta(\omega^* + \omega)$ for suitable η. Furthermore there will be elements $n\,u$ (for each natural number n.) and no two of these can be equal since $nu = mu$ implies $u = 0$ contrary to assumption that u was not a natural number. Still further we find new elements of the forms u^2, u^u, etc. Thus the admission of a single new element automatically involves an extended hierarchy of elements.

We may note that although all non-standard models for number theory contain infinite desceding sequences $(\ldots, u-2, u-1, u,)$ The formula which expresses that "every set of number has a least" is true (and in fact provable), a situation made possible by reinterpreting "set". We may describe this state of affairs by saying that the property of being a well-ordering relation is not <u>absolute.</u>

By contrast a property such as to be infinite (for sets of individuals, say,) is absolute in the sense that if $A_0(\alpha)$ denotes this property (under standard interpretation) and $A(\alpha), B\alpha$ is provable (for some closed $B\alpha$) Then the denotation of $B\alpha$ with respect to every general model has this property. If in some general model there exists a one-one correspondence between a given set and a proper subset of itself that set will be infinite. Notice, however, that if no such one-one correspondence exists it may only be because all such correspondences have been excluded from the domain (of appropriate type) of that particular model, while the set may still be infinite. Thus the property to be a finite set is not absolute, although the properties of being infinite or having any particular finite

number of elements, are. This is a natural correlate of the fact that the set of finite cardinals cannot be fully characterized.

What are the implications of our viewpoint as regards Gödel's demonstration of the impossibility of a consistency-proof for certain logics? We may analyze Gödel's procedure into three components: The construction of a certain formula, Wid, the demonstration that Wid expresses the proposition that the logic is consistent - when the notation of the calculus is interpreted as referring to any standard model, - and finally the proof that Wid is not a formal theorem if the system is consistent.

It follows from our completeness proof that Wid denotes F for some general model. It would appear a priori possible, however, for some other formula to express the consistency of the formal system and at the same time denote T for every model (thus being provable). Whether such a procedure, if found, would be acceptable as a genuine consistency proof would depend in part on the manner in which it was shown that the formula involved did denote T just in case the logic were consistent. We thus emphasize that the description of what constitutes a proof of a proposition requires the specification not only of formal rules for manipulating a particular symbolism, but rules detailing in what ways the symbolism is to be interpreted.

As a result of this study we tend to reinterpret Gödel's incompleteness result as asserting not primarily a limitation on our ability to prove but rather on our ability to specify what we mean (to what we are referring) when we use a symbolic system in accordance with recursive rules. If we reflect on the way in which we learn to use the symbolism

of our natural languages for the communication of mathematical ideas it seems not implausible that even in that case there is the possibility of differing interpretation coexisting with common rules of manipulation, and that failure to realize what non-standard meanings are consistent with our rules of use blocks our ability to establish results. But perhaps this is philosophy.

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